Combinatorial models and bijections in Parabolic Cataland, type A and B

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With Cesar Ceballos and Henri Mühle, arXiv:1903.08515
With Henri Mühle et Jean-Christophe Novelli, partly in progress, arXiv:2112.13400

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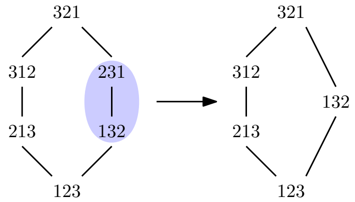
Tamari lattice, as quotient of the weak order

$\mathfrak{S}_n$ as a Coxeter group generated by $s_i = (i, i+1)$

For $w \in \mathfrak{S}_n$, $\ell(w) =$ min. length of factorization of w in s_i

(Left) weak order $\leq_{\text{weak}}$: $s_i w$ covers w iff $\ell(s_i w) = \ell(w) + 1$



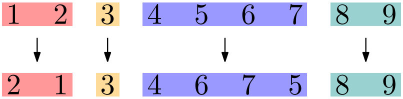
Sylvester class : permutations with the same binary search tree

Only one 231-avoiding in each class. Induced order = Tamari.

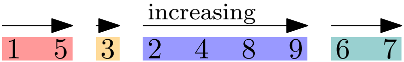
Parabolic subgroup and parabolic quotient of $\mathfrak{S}_n$

Parabolic subgroup : $\langle s_j, j \in J\rangle$ for $J \subseteq [n-1]$

Has the form $\mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_k}$ with $\alpha = (\alpha_1, \ldots, \alpha_k)$ a composition of n.



Parabolic quotient : $\mathfrak{S}_n^\alpha = \mathfrak{S}_n / (\mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_k})$.



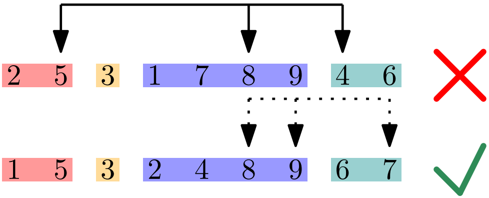
Increasing in each region

Parabolic permutations avoiding 231

$(\alpha, 231)$-pattern : indices $i < j < k$ in different regions with

- $w(k) < w(i) < w(j)$,
- $w(k) + 1 = w(i)$.

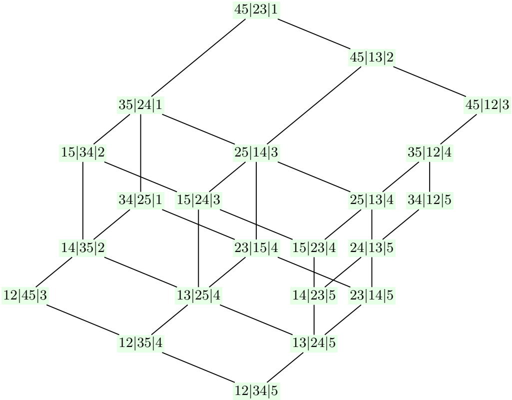
$(\alpha, 231)$-avoiding permutations: without $(\alpha, 231)$ patterns



$\mathfrak{S}_n^{\alpha}(231)$: set of $(\alpha, 231)$-avoiding permutations

Parabolic Tamari lattice

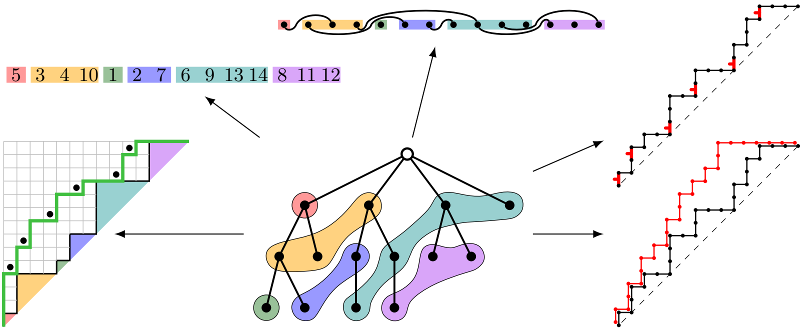
Parabolic Tamari lattice $\mathcal{T}_n^\alpha = (\mathfrak{S}_n^\alpha(231), \leq_{\mathsf{weak}})$ (Mühle–Williams 2019)



Isomorphic to certain ν-Tamari lattices (Ceballos–F.–Mühle 2020, F.–Mühle–Novelli 2021).

Parabolic Cataland

Ceballos–F.–Mühle 2020: a combinatorial model as center of bijections!



- Simplifying some bijections in (Mühle–Williams 2019).
- Link to walks in the quadrant in (Bousquet-Mélou–Mishna 2010).
- Solving a conjecture in (Bergeron–Ceballos–Pilaud 2022).
- Recovering the zeta map in q,t-Catalan combinatorics.

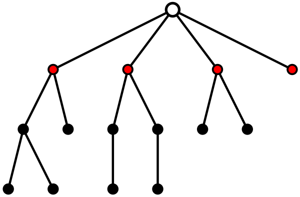
Left-aligned colored trees

- T : plane tree with n non-root nodes;
- $\alpha = (\alpha_1, \ldots, \alpha_k)$: composition of n

Active nodes : not yet colored, but parent is colored or the root.

Coloring algorithm : For i from 1 to k,

- Fail if there are less than α_i active nodes;
- Otherwise, color the first α_i from left to right with color i.



$$\alpha = (\mathbf{1}, 3, 1, 2, 4, 3) \vdash 14$$

When succeeded, (T, α) is a left-aligned colored tree (or a LAC tree).

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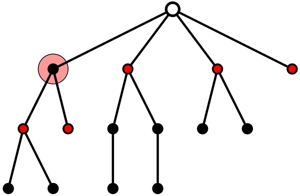

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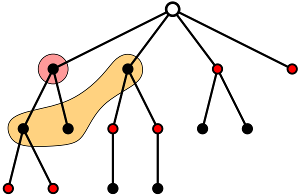

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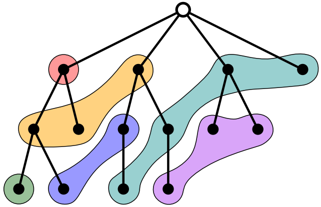

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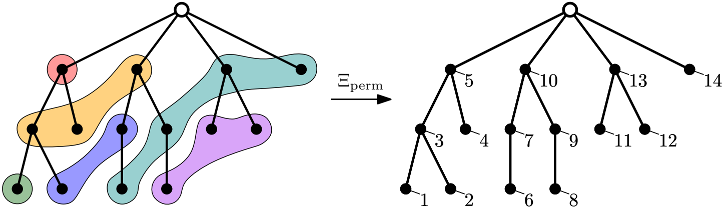
When succeeded, (T, α) is a left-aligned colored tree (or a LAC tree).

To permutations

Label from n to 1 clockwise, then read by regions.



(T, α)

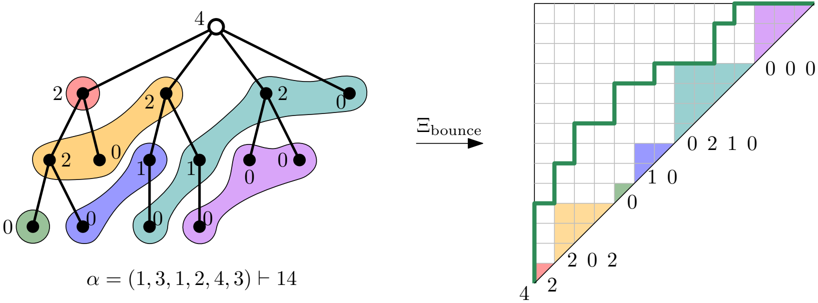
$\Xi_{\text{perm}}(T, \alpha) = 5 \mid 3\ 4\ 10 \mid 1 \mid 2\ 7 \mid 6\ 9\ 13\ 14 \mid 8\ 11\ 12 \in \mathfrak{S}_n^{\alpha}(231)$

A variant of binary search tree.

To bounce pairs

Bounce pair: A Dyck path P above the bounce path of composition α.

Based on the root poset of $\mathfrak{S}_n$ (Mühle–Williams 2019).



#↑ on $x = 0$ ⇔ #children of the root

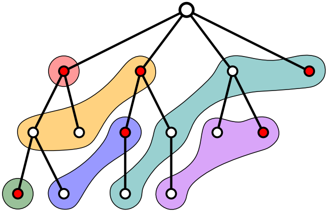
#↑ in region k ⇔ #children of nodes in region k from right to left

To steep pairs

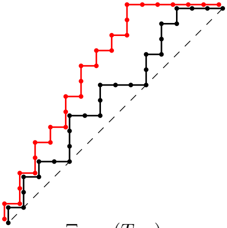
Steep pair: Two nested Dyck paths, the upper one without $\rightarrow\rightarrow$ except the end (Bergeron–Ceballos–Pilaud 2022, Hopf algebra on pipe dreams).

In bijection with walks in $\mathbb{N}^2$ from $(0,0)$ to x-axis with steps $\{\uparrow, \nwarrow, \searrow\}$. (Bousquet-Mélou–Mishna 2010, Mishna–Rechnitzer, 2009)

Asymptotics $cn^{-1/2}3^n$, but not D-finite...



(T, α)

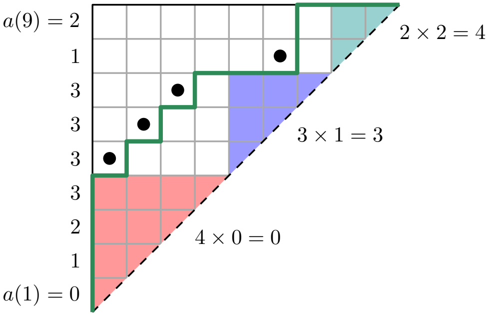


$\Xi_{\text{steep}}(T, \alpha)$

- Lower path: depth-first search from right to left.
- Upper path: red node to $\uparrow$, white node to $\rightarrow\uparrow$, padding with $\rightarrow$.

All α of n combined!

Detour to q, t-Catalan combinatorics



$\text{bounce}(D) = \sum_i (i-1)\alpha_i = 7$

$\text{area}(D) = \sum_i a(i) = 18$

$\text{dinv}(D) = \#\{(i,j) \mid i < j, (a(i) = a(j) \vee a(i) = a(j) + 1\} = 13$

- $\text{bounce}(P)$: sum of $(i-1)\alpha_i$, with α constructed greedily.
- $\text{area}(P)$: number of squares under P.
- $\text{dinv}(P)$: complicated...

Zeta map from diagonal harmonics

Theorem (Haglund and Haiman, see Haglund 2008)

By summing over all Dyck paths of order n, we have

$$\sum_{n\geq 0} z^n \sum_{D\in\mathcal{D}_n} q^{\mathrm{area}(D)} t^{\mathrm{bounce}(D)} = \sum_{n\geq 0} z^n \sum_{D\in\mathcal{D}_n} q^{\mathrm{dinv}(D)} t^{\mathrm{area}(D)}.$$

Related to diagonal coinvariant space.

Also symmetry in q, t by algebraic argument only.

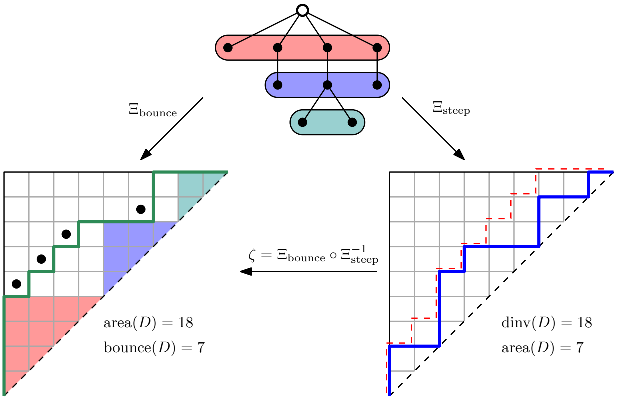
Theorem (Haglund 2008)

There is a bijection ζ on Dyck paths that transfers the pairs of statistics

$$(\mathrm{dinv}, \mathrm{area}) \to (\mathrm{area}, \mathrm{bounce}).$$

First given in (Andrews, Krattenthaler, Orsina and Papi, 2001).

Zeta map, via LAC trees



- area-dinv: # certain pairs of nodes
- bounce-area: sum of (depth - 1) over all nodes
- Also the labelled version in (Haglund–Loehr 2005)
- A bijective proof of Conj. 7.1 in (Matherne–Morales–Selover 2022)

Coxeter group, type B

Type B: permutations w of $\pm[n] \stackrel{\text{def}}{=} \{-n, \ldots, -1, 1, \ldots, n\}$ that are sign-symmetric, *i.e.*, $w(-i) = -w(i)$. Also hyperoctahedral group $\mathfrak{H}_n$.

One-line notation: (with $\overline{k}$ for $-k$)

$$w = \overline{9}\,\overline{7}\,\overline{8}\,\overline{5}\,\overline{6}\,1\,\overline{3}\,\overline{4}\,2 \mid \overline{2}\,4\,3\,\overline{1}\,6\,5\,8\,7\,9.$$

Or only the right (positive) part: $w =\mid \overline{2}\,4\,3\,\overline{1}\,6\,5\,8\,7\,9$

Inversion of $w \in \mathfrak{H}_n$: indices $i, j \in \pm[n]$ with $i < j$ but $w(i) > w(j)$

Sign-symmetry $\Rightarrow$ if i, j is an inversion, then $-j, -i$ too.

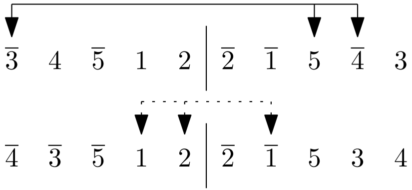
Weak order (left): $w \leq_{\text{weak}} w' \Leftrightarrow$ inversion set of w' includes that of w

Tamari lattice, type B

Successor in $\pm[n]$: $i^+ = i + 1$, except $(-1)^+ = 1$

Type-B 231-pattern in w: indices $i < j < k$ in $\pm[n]$ such that

- $j > 0$; (to break sign-symmetry)
- $w(i) = w(k)^+$, $w(j) > w(i)$.



$\mathfrak{H}_n(231)$: 231-avoiding sign-symmetric permutations

Type-B Tamari lattice (Reading 2007): $\mathrm{Tam}_B(n) \stackrel{\text{def}}{=} (\mathfrak{H}_n(231), \leq_{\text{weak}})$.

Parabolic quotient of $\mathfrak{H}_n$

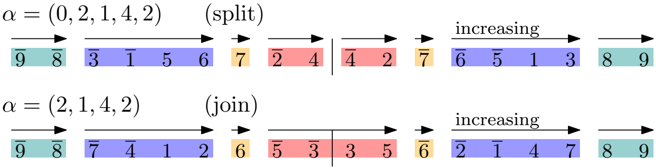
Generators: $S = \{s_0, s_1, \ldots, s_{n-1}\}$

- For $i \geq 1$, s_i exchanges i and $i+1$ (thus $-i$ and $-i-1$);
- s_0 exchanges 1 and -1.

Type-B composition: $\alpha = (\alpha_1, \ldots, \alpha_k)$, with possibly $\alpha_0 = 0$

Split when α starts with 0, join otherwise.

Parabolic quotient of $\mathfrak{H}_n$, denoted by $\mathfrak{H}_\alpha$



In the join case, the central region is positive for positive indices.

Type-B $(\alpha, 231)$-patterns

Type-B $(\alpha, 231)$-pattern in w: indices $i < j < k$ in $\pm[n]$ such that

- i, j, k in different regions; (parabolic)
- $j > 0$; (to break sign-symmetry)
- $w(i) = w(k)^+$;
- $w(j) > w(i)$ when α is split or $j > \alpha_1$; (231)
- $w(j) < w(k)$ when α is join and $j \leq \alpha_1$. (312)

Split case:

Pattern $4 \mid \overline{7}\ \overline{3} \mid \text{\textcircled{1}} \mid \overline{6}\ \overline{2}\ 5 \mid \overline{5}\ 2\ \text{\textcircled{6}} \mid \overline{\text{\textcircled{1}}} \mid 3\ 7 \mid \overline{4}$

Pattern $4 \mid \overline{5}\ \overline{1} \mid 6 \mid \overline{2}\ 3\ 7 \mid \overline{7}\ \text{\textcircled{3}}\ 2 \mid \overline{6} \mid 1\ \text{\textcircled{5}} \mid \overline{\text{\textcircled{4}}}$

Join case:

Not pattern $\overline{\text{\textcircled{6}}}\ \overline{4} \mid \overline{8}\ 7 \mid \overline{5}\ \overline{3}\ \overline{2}\ \overline{1}\ \text{\textcircled{1}}\ 2\ 3\ 5 \mid \overline{\text{\textcircled{7}}}\ 8 \mid 4\ 6$

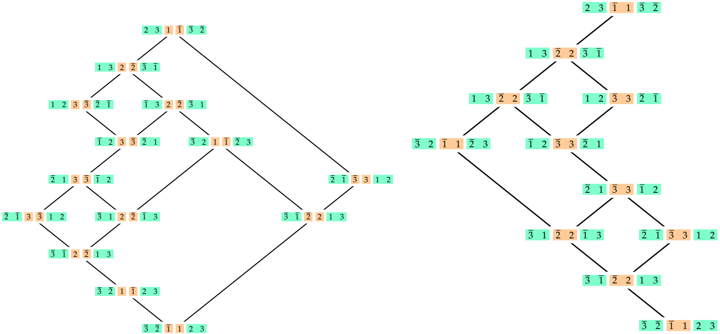
Pattern $\overline{7}\ \text{\textcircled{5}} \mid \overline{4}\ 3 \mid \overline{8}\ \overline{6}\ \overline{2}\ \overline{1}\ 1\ \text{\textcircled{2}}\ 6\ 8 \mid \overline{3}\ \text{\textcircled{4}} \mid \overline{5}\ 7$

Flipped for the joined region!

Type-B parabolic Tamari lattice

$\mathfrak{H}_\alpha(231)$: Type-B $(\alpha, 231)$-avoiding permutations

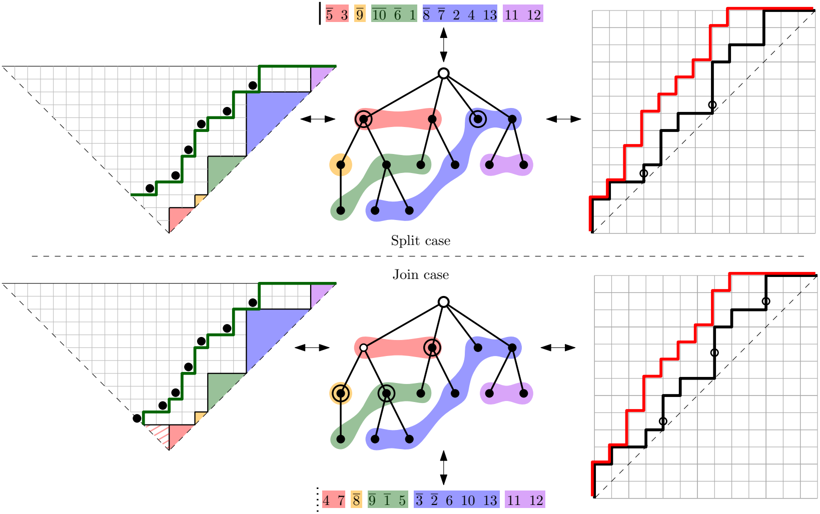
Type-B parabolic Tamari lattice: $\mathsf{Tam}_B(\alpha) = (\mathfrak{H}_\alpha(231), \leq_{\mathsf{weak}})$ How?



Theorem (F.–Mühle-Novelli 2022+)

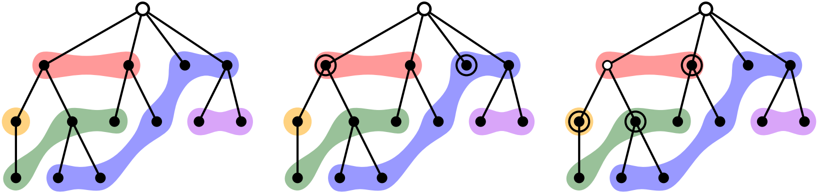
$\mathsf{Tam}_B(\alpha)$ *is a quotient lattice of the weak order of* $\mathfrak{H}_\alpha$*, and is congruence uniform and trim.*

Combinatorial models



Work in progress. Some bijections clear, some less.

LAC trees, type B



$\alpha = (2, 1, 3, 5, 2)$
or $\alpha = (0, 2, 1, 3, 5, 2)$

$\alpha = (0, 2, 1, 3, 5, 2)$

$\alpha = (2, 1, 3, 5, 2)$

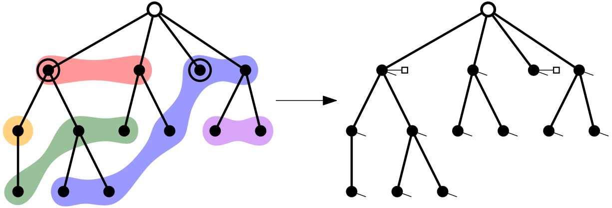
Type-B LAC tree: LAC tree (T, α) + switch nodes among

- (α split) children of the root;
- (α join) nodes in region 1 + chidren of the first child of the root.

Moreover, for α join, at most half of the switch nodes are in region 1.

For α join, first child of the root acts as a second root.

To permutation



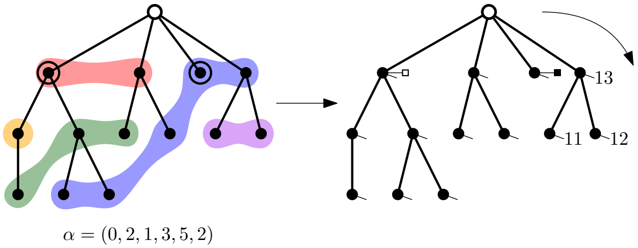
$\alpha = (0, 2, 1, 3, 5, 2)$

Alternating contour walk:

- Label nodes from n to 1, with sign given by direction;
- Switch on switch buds (squares).

Same for the join case!

To permutation



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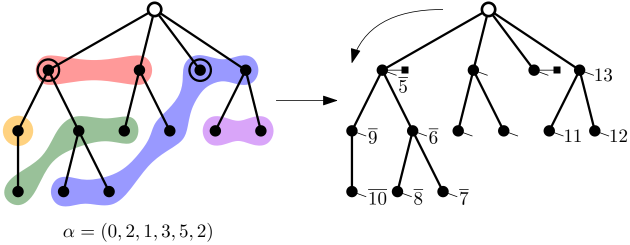

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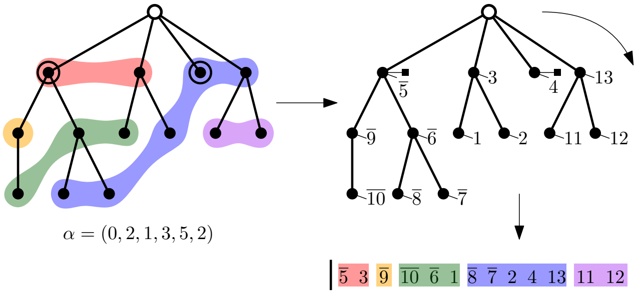

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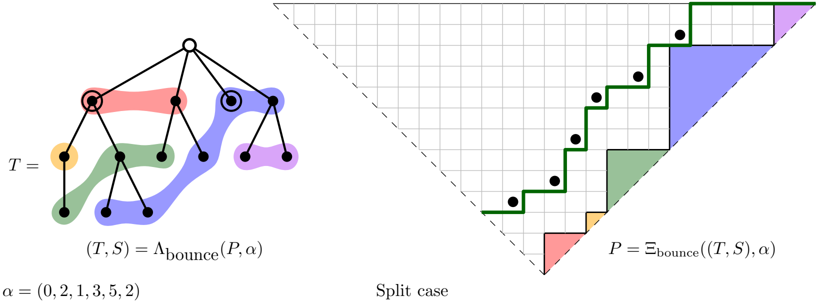
To domain paths (type-B bounce pairs)

Domain based on the root poset of type-B.



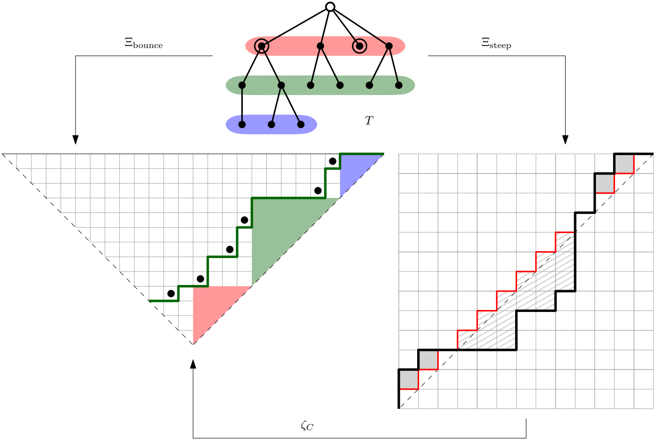
Split case:

- Right part: just like in type A
- Left part: given by paired up switch nodes, counted from right to left

Join case: an extra forbidden region, slightly more complicated... What?

Type-C zeta map

Sulzgruber–Thiel 2018: (labelled) Zeta map for type B, C and D



Steep pair replaced by box path for the alternating contour walk.

We recover (labelled) zeta map for type C. Also transfer dinv ↔ area.

Some enumerative theorems

Cover inversion of $w \in \mathfrak{H}_n$: inversion $i < j$ with $w(i) = w(j)^+$.

cov(w): #cover inversions of w.

Proposition (F.–Mühle–Novelli 2022+)

Take $c_\alpha(x) = \sum_{w\in\mathfrak{H}_\alpha(231)} x^{\mathrm{cov}(w)}$. *Then for* $\alpha = (t, 1, \ldots, 1)$, *we have*

$$c_\alpha(x) = \sum_{k=0}^{n-t} \binom{n-t}{k}\binom{n+t}{k} x^k, \quad |\mathfrak{H}_\alpha(231)| = c_\alpha(1) = \binom{2n}{n-t}.$$

Cover inversions = valleys in bounce path

Proposition (F.–Mühle–Novelli 2022+)

For $\alpha = (0, 1, 1, \ldots, 1, 2)$, $|\mathfrak{H}_\alpha(231)|$ *is the type-D Catalan number:*

$$|\mathfrak{H}_\alpha(231)| = \frac{3n-2}{n}\binom{2n-2}{n-1}.$$

Further directions

- Combinatorial description of the order?
- Link to possible type-B ν-Tamari (Ceballos–Padrol–Sarmiento '19)?
- Type-B q, t-Catalan statistics (Stump 2010)?
- Type-B zeta map?
- Enumeration? (Lattice path model known for split case)
- Type-D parabolic Cataland?

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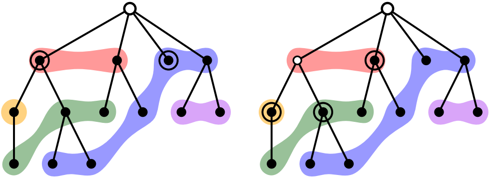
Thank you for your attention!



Where are the conditions from?

Reading 2007: Universal construction of Tamari (Cambrian) lattices for all type

On c-aligned elements, with c a Coxeter element (product of all s_i)

Type B: we take $c = s_0 s_1 \cdots s_{n-1}$

w is c-aligned $\Leftrightarrow$ forcing relations: some $t \in \mathsf{cov}(w) \Rightarrow$ some $s \in \mathsf{Inv}(w)$

Determined by a linear order of inversions given by the c-sorting word of the longest element in $\mathfrak{H}_n$

Type B, **parabolic**: replace the longest element in $\mathfrak{H}_n$ by that in $\mathfrak{H}_\alpha$

A slide not meant to be read

!!! Headache warning !!!

$w \in \mathfrak{H}_\alpha$ is c-aligned if, for all $1 \leq i < k \leq n$,

(1) if $[\![i]\!] \in \mathrm{cov}(w)$, then $[\![j]\!] \in \mathrm{Inv}(w)$ for all $1 \leq j < i$ with i, j in different regions;

(2) if $(\!(i\ k)\!) \in \mathrm{cov}(w)$, then $(\!(i\ j)\!) \in \mathrm{Inv}(w)$ such that i, j, k are in different regions;

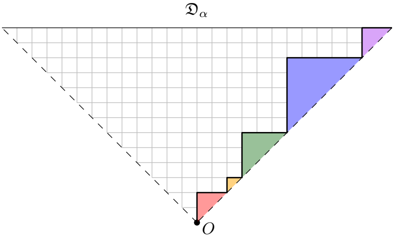
(3) if $(\!(-k\ i)\!) \in \mathrm{cov}(w)$, then

- (3a) $[\![i]\!] \in \mathrm{Inv}(w)$ when $i > \alpha_1$ or α is split,
- (3b) $(\!(-j\ i)\!) \in \mathrm{Inv}(w)$ for $1 \leq j < k$ with j, k in different regions when α is split or $j > \alpha_1$,
- (3c) $(\!(j\ k)\!) \in \mathrm{Inv}(w)$ when $j \leq \alpha_1$, $j \neq i$ and α is join,
- (3d) $(\!(-k\ j)\!) \in \mathrm{Inv}(w)$ for $1 \leq j < i$ with i, j in different regions when α is split or $j > \alpha_1$,
- (3e) $(\!(j\ i)\!) \in \mathrm{Inv}(w)$ when $i > j > \alpha_1$ and α is join.

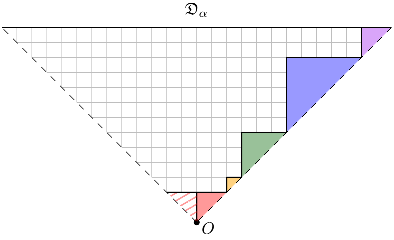
Summed up nicely by pattern avoidance !

> Back <

Domains for type-B compositions



$\alpha = (0, 2, 1, 3, 5, 2)$



$\alpha = (2, 1, 3, 5, 2)$

For the join case, exchange (?) the roles of the root and the second root.

- Highest y on $x = 0$: $\alpha_1 + \#$ children of the second root.
- $\# \uparrow$ on $x = \alpha_1$: $\#$ children of the root not in region 1.

So that the highest y-coordinate on $x = 0$ is the max number of switch nodes.

> Back <