Combinatorial models and bijections in Parabolic Cataland, type A and B

Wenjie Fang, LIGM, Université Gustave Eiffel

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**Tamari lattice, as quotient of the weak order**

$\mathcal{S}_n$ as a Coxeter group generated by $s_i = (i, i + 1)$

For $w \in \mathcal{S}_n$, $\ell(w) = \text{min. length of factorization of } w \text{ in } s_i$

**Left weak order** $\leq_{\text{weak}} : s_i w \text{ covers } w \iff \ell(s_i w) = \ell(w) + 1$

![Tamari lattice diagram]

Sylvester class: permutations with the same binary search tree

Only one 231-avoiding in each class. Induced order = Tamari.
Parabolic subgroup and parabolic quotient of $\mathfrak{S}_n$

**Parabolic subgroup**: $\langle s_j, j \in J \rangle$ for $J \subseteq [n - 1]$

Has the form $\mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_k}$ with $\alpha = (\alpha_1, \ldots, \alpha_k)$ a composition of $n$.

Parabolic quotient: $\mathfrak{S}_n^\alpha = \mathfrak{S}_n / (\mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_k})$.

Increasing in each region
(\(\alpha, 231\))-pattern: indices \(i < j < k\) in different regions with
- \(w(k) < w(i) < w(j)\),
- \(w(k) + 1 = w(i)\).

(\(\alpha, 231\))-avoiding permutations: without (\(\alpha, 231\)) patterns

\[\mathcal{S}^\alpha_n(231) : \text{set of } (\alpha, 231)-\text{avoiding permutations}\]
Parabolic Tamari lattice

Parabolic Tamari lattice $T_n^\alpha = (\mathcal{G}_n^\alpha (231), \leq_{\text{weak}})$ (Mühle–Williams 2019)

Parabolic Cataland

Ceballos–F.–Mühle 2020: a combinatorial model as center of bijections!

- Simplifying some bijections in (Mühle–Williams 2019).
- Link to walks in the quadrant in (Bousquet-Mélou–Mishna 2010).
- Solving a conjecture in (Bergeron–Ceballos–Pilaud 2022).
- Recovering the zeta map in $q, t$-Catalan combinatorics.
Left-aligned colored trees

- $T$: plane tree with $n$ non-root nodes;
- $\alpha = (\alpha_1, \ldots, \alpha_k)$: composition of $n$

Active nodes: not yet colored, but parent is colored or the root.

**Coloring algorithm**: For $i$ from 1 to $k$,
- Fail if there are less than $\alpha_i$ active nodes;
- Otherwise, color the first $\alpha_i$ from left to right with color $i$.

$$\alpha = (1, 3, 1, 2, 4, 3) \vdash 14$$

When succeeded, $(T, \alpha)$ is a left-aligned colored tree (or a LAC tree).
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To permutations

Label from $n$ to 1 clockwise, then read by regions.

$\Xi_{\text{perm}}(T, \alpha) = 5 \mid 3 4 10 \mid 1 \mid 2 7 \mid 6 9 13 14 \mid 8 11 12 \in S_n(231)$

A variant of binary search tree.
To bounce pairs

**Bounce pair**: A Dyck path $P$ above the bounce path of composition $\alpha$.

Based on the root poset of $\mathcal{S}_n$ (Mühle–Williams 2019).

$$\alpha = (1, 3, 1, 2, 4, 3) \vdash 14$$

#↑ on $x = 0 \Leftrightarrow$ #children of the root

#↑ in region $k \Leftrightarrow$ #children of nodes in region $k$ from right to left
To steep pairs

**Steep pair** : Two nested Dyck paths, the upper one without $\rightarrow\rightarrow$ except the end (Bergeron–Ceballos–Pilaud 2022, Hopf algebra on pipe dreams).

In bijection with walks in $\mathbb{N}^2$ from $(0,0)$ to $x$-axis with steps $\{ \uparrow, \nwarrow, \swarrow \}$. (Bousquet-Mélou–Mishna 2010, Mishna–Rechnitzer, 2009)

Asymptotics $cn^{-1/2}3^n$, but not D-finite...

- Lower path: depth-first search from right to left.
- Upper path: red node to $\uparrow$, white node to $\rightarrow\uparrow$, padding with $\rightarrow$.

All $\alpha$ of $n$ combined!
Detour to $q,t$-Catalan combinatorics

\[
\begin{align*}
\text{bounce}(D) &= \sum_i (i-1)\alpha_i = 7 \\
\text{area}(D) &= \sum_i a(i) = 18 \\
\text{dinv}(D) &= \#\{(i,j) \mid i < j, (a(i) = a(j) \lor a(i) = a(j) + 1)\} = 13
\end{align*}
\]

- **bounce**($P$): sum of $(i-1)\alpha_i$, with $\alpha$ constructed greedily.
- **area**($P$): number of squares under $P$.
- **dinv**($P$): complicated...
Zeta map from diagonal harmonics

Theorem (Haglund and Haiman, see Haglund 2008)

By summing over all Dyck paths of order $n$, we have

$$
\sum_{n \geq 0} z^n \sum_{D \in \mathcal{D}_n} q^{\text{area}(D)} t^{\text{bounce}(D)} = \sum_{n \geq 0} z^n \sum_{D \in \mathcal{D}_n} q^{\text{dinv}(D)} t^{\text{area}(D)}.
$$

Related to diagonal coinvariant space.

Also symmetry in $q, t$ by algebraic argument only.

Theorem (Haglund 2008)

There is a bijection $\zeta$ on Dyck paths that transfers the pairs of statistics

$$(\text{dinv}, \text{area}) \rightarrow (\text{area}, \text{bounce}).$$

First given in (Andrews, Krattenthaler, Orsina and Papi, 2001).
Zeta map, via LAC trees

\[
\zeta = \Xi_{\text{bounce}} \circ \Xi_{\text{steep}}^{-1}
\]

- **area-dinv**: \# certain pairs of nodes
- **bounce-area**: sum of \(\text{depth} - 1\) over all nodes
- Also the labelled version in (Haglund–Loehr 2005)
- A bijective proof of Conj. 7.1 in (Matherne–Morales–Selover 2022)
Coxeter group, type B

**Type B**: permutations $w$ of $\pm[n] \overset{\text{def}}{=} \{-n, \ldots, -1, 1, \ldots, n\}$ that are sign-symmetric, i.e., $w(-i) = -w(i)$. Also hyperoctahedral group $\mathfrak{S}_n$.

One-line notation: (with $\bar{k}$ for $-k$)

$$w = \bar{9} \bar{7} \bar{8} \bar{5} \bar{6} \ 1 \ 3 \ 4 \ 2 \ | \ 2 \ 4 \ 3 \ 1 \ 6 \ 5 \ 8 \ 7 \ 9.$$

Or only the right (positive) part: $w = | \ 2 \ 4 \ 3 \ 1 \ 6 \ 5 \ 8 \ 7 \ 9$

Inversion of $w \in \mathfrak{S}_n$: indices $i, j \in \pm[n]$ with $i < j$ but $w(i) > w(j)$

Sign-symmetry $\Rightarrow$ if $i, j$ is an inversion, then $-j, -i$ too.

**Weak order** (left): $w \leq_{\text{weak}} w' \iff$ inversion set of $w'$ includes that of $w$
Tamari lattice, type B

Successor in $\pm[n]$: $i^+ = i + 1$, except $(-1)^+ = 1$

Type-B 231-pattern in $w$: indices $i < j < k$ in $\pm[n]$ such that
- $j > 0$; (to break sign-symmetry)
- $w(i) = w(k)^+$, $w(j) > w(i)$.

$\mathcal{H}_n(231)$: 231-avoiding sign-symmetric permutations

Type-B Tamari lattice (Reading 2007): $\text{Tam}_B(n) \overset{\text{def}}{=} (\mathcal{H}_n(231), \leq_{\text{weak}})$. 
Parabolic quotient of $\tilde{H}_n$

Generators: $S = \{s_0, s_1, \ldots, s_{n-1}\}$
- For $i \geq 1$, $s_i$ exchanges $i$ and $i + 1$ (thus $-i$ and $-i - 1$);
- $s_0$ exchanges 1 and $-1$.

Type-B composition: $\alpha = (\alpha_1, \ldots, \alpha_k)$, with possibly $\alpha_0 = 0$

Split when $\alpha$ starts with 0, join otherwise.

Parabolic quotient of $\tilde{H}_n$, denoted by $H_\alpha$

$\alpha = (0, 2, 1, 4, 2)$ (split)

\[
\begin{array}{ccccccc}
\scriptsize{9} & \scriptsize{8} & \scriptsize{3} & \scriptsize{1} & \scriptsize{5} & \scriptsize{6} & \scriptsize{7} \\
\Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow \\
\scriptsize{9} & \scriptsize{8} & \scriptsize{3} & \scriptsize{1} & \scriptsize{5} & \scriptsize{6} & \scriptsize{7} \\
\end{array}
\]

$\alpha = (2, 1, 4, 2)$ (join)

\[
\begin{array}{ccccccc}
\scriptsize{9} & \scriptsize{8} & \scriptsize{7} & \scriptsize{4} & \scriptsize{1} & \scriptsize{2} & \scriptsize{6} \\
\Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow \\
\scriptsize{9} & \scriptsize{8} & \scriptsize{7} & \scriptsize{4} & \scriptsize{1} & \scriptsize{2} & \scriptsize{6} \\
\end{array}
\]

Increasing

$\alpha = (0, 2, 1, 4, 2)$ (split)

\[
\begin{array}{ccccccc}
\scriptsize{9} & \scriptsize{8} & \scriptsize{3} & \scriptsize{1} & \scriptsize{5} & \scriptsize{6} & \scriptsize{7} \\
\Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow \\
\scriptsize{6} & \scriptsize{5} & \scriptsize{1} & \scriptsize{3} & \scriptsize{7} & \scriptsize{8} & \scriptsize{9} \\
\end{array}
\]

$\alpha = (2, 1, 4, 2)$ (join)

\[
\begin{array}{ccccccc}
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\Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow \\
\scriptsize{2} & \scriptsize{1} & \scriptsize{4} & \scriptsize{7} & \scriptsize{8} & \scriptsize{9} \\
\end{array}
\]

In the join case, the central region is positive for positive indices.
Type-B \((\alpha, 231)\)-patterns

**Type-B \((\alpha, 231)\)-pattern** in \(w\): indices \(i < j < k\) in \(\pm[n]\) such that

- \(i, j, k\) in different regions; \hspace{1em} (parabolic)
- \(j > 0\); \hspace{1em} (to break sign-symmetry)
- \(w(i) = w(k)^+\);
- \(w(j) > w(i)\) when \(\alpha\) is split or \(j > \alpha_1\); \hspace{1em} (231)
- \(w(j) < w(k)\) when \(\alpha\) is join and \(j \leq \alpha_1\). \hspace{1em} (312)

Split case:

<table>
<thead>
<tr>
<th>Pattern</th>
<th>4 7 3 1 6 2 5 5 2 6 1 3 7 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pattern</td>
<td>4 5 1 6 2 3 7 7 3 2 6 1 5 4</td>
</tr>
</tbody>
</table>

Join case:

<table>
<thead>
<tr>
<th>Not pattern</th>
<th>6 4 8 7 5 3 2 1 1 2 3 5 7 8 4 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pattern</td>
<td>7 5 4 3 8 6 2 1 1 2 6 8 3 4 5 7</td>
</tr>
</tbody>
</table>

Flipped for the joined region!
Type-B parabolic Tamari lattice

$\mathcal{S}_\alpha(231)$: Type-B $(\alpha, 231)$-avoiding permutations

Type-B parabolic Tamari lattice: $\text{Tam}_B(\alpha) = (\mathcal{S}_\alpha(231), \leq_{\text{weak}})$

How?

Theorem (F.-Mühle-Novelli 2022+)

$\text{Tam}_B(\alpha)$ is a quotient lattice of the weak order of $\mathcal{S}_\alpha$, and is congruence uniform and trim.
Combinatorial models

Work in progress. Some bijections clear, some less.
**LAC trees, type B**

\[
\alpha = (2, 1, 3, 5, 2) \\
\text{or } \alpha = (0, 2, 1, 3, 5, 2)
\]

**Type-B LAC tree:** LAC tree \((T, \alpha)\) + switch nodes among
- \((\alpha \text{ split})\) children of the root;
- \((\alpha \text{ join})\) nodes in region 1 + children of the first child of the root.

Moreover, for \(\alpha \text{ join}\), at most half of the switch nodes are in region 1.

For \(\alpha \text{ join}\), first child of the root acts as a second root.
To permutation

\[ \alpha = (0, 2, 1, 3, 5, 2) \]

Alternating contour walk:

- Label nodes from \( n \) to 1, with sign given by direction;
- Switch on switch buds (squares).

Same for the join case!
To permutation

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- Label nodes from \( n \) to 1, with sign given by direction;
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Same for the join case!
To domain paths (type-B bounce pairs)

Domain based on the root poset of type-B.

\[ T = \Lambda_{\text{bounce}}(P, \alpha) \]

\[ (T, S) = \Lambda_{\text{bounce}}(P, \alpha) \]

\[ \alpha = (0, 2, 1, 3, 5, 2) \]

Split case:
- **Right part**: just like in type A
- **Left part**: given by paired up switch nodes, counted from right to left

Join case: an extra forbidden region, slightly more complicated...

What?
Type-C zeta map

Sulzgruber–Thiel 2018: (labelled) Zeta map for type B, C and D

Steep pair replaced by box path for the alternating contour walk.

We recover (labelled) zeta map for type C. Also transfer \( \text{dinv} \leftrightarrow \text{area} \).
Some enumerative theorems

**Cover inversion** of \( w \in \mathcal{H}_n \): inversion \( i < j \) with \( w(i) = w(j)^+ \).

\[ \text{cov}(w): \#\text{cover inversions of } w. \]

**Proposition (F.–Mühle–Novelli 2022+)**

Take \( c_{\alpha}(x) = \sum_{w \in \mathcal{H}_\alpha(231)} x^{\text{cov}(w)} \). Then for \( \alpha = (t, 1, \ldots, 1) \), we have

\[ c_{\alpha}(x) = \sum_{k=0}^{n-t} \binom{n-t}{k} \binom{n+t}{k} x^k, \quad |\mathcal{H}_\alpha(231)| = c_{\alpha}(1) = \binom{2n}{n-t}. \]

Cover inversions = valleys in bounce path

**Proposition (F.–Mühle–Novelli 2022+)**

For \( \alpha = (0, 1, 1, \ldots, 1, 2) \), \( |\mathcal{H}_\alpha(231)| \) is the type-D Catalan number:

\[ |\mathcal{H}_\alpha(231)| = \frac{3n-2}{n} \binom{2n-2}{n-1}. \]
Further directions

- Combinatorial description of the order?
- Link to possible type-B $\nu$-Tamari (Ceballos–Padrol–Sarmiento ’19)?
- Type-B $q, t$-Catalan statistics (Stump 2010)?
- Type-B zeta map?
- Enumeration? (Lattice path model known for split case)
- Type-D parabolic Cataland?
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Thank you for your attention!
Where are the conditions from?

Reading 2007: Universal construction of Tamari (Cambrian) lattices for all type

On $c$-aligned elements, with $c$ a Coxeter element (product of all $s_i$)

Type B: we take $c = s_0 s_1 \cdots s_{n-1}$

$w$ is $c$-aligned $\iff$ forcing relations: some $t \in \text{cov}(w) \Rightarrow$ some $s \in \text{Inv}(w)$

Determined by a linear order of inversions given by the $c$-sorting word of the longest element in $\mathcal{H}_n$

Type B, parabolic: replace the longest element in $\mathcal{H}_n$ by that in $\mathcal{H}_\alpha$
$w \in \mathcal{H}_\alpha$ is $c$-aligned if, for all $1 \leq i < k \leq n$,

1. if $[i] \in \text{cov}(w)$, then $[j] \in \text{Inv}(w)$ for all $1 \leq j < i$ with $i, j$ in different regions;
2. if $(i k) \in \text{cov}(w)$, then $(i j) \in \text{Inv}(w)$ such that $i, j, k$ are in different regions;
3. if $((-k i)) \in \text{cov}(w)$, then
   - $[i] \in \text{Inv}(w)$ when $i > \alpha_1$ or $\alpha$ is split,
   - $((-j i)) \in \text{Inv}(w)$ for $1 \leq j < k$ with $j, k$ in different regions when $\alpha$ is split or $j > \alpha_1$,
   - $((j k)) \in \text{Inv}(w)$ when $j \leq \alpha_1$, $j \neq i$ and $\alpha$ is join,
   - $((-k j)) \in \text{Inv}(w)$ for $1 \leq j < i$ with $i, j$ in different regions when $\alpha$ is split or $j > \alpha_1$,
   - $((j i)) \in \text{Inv}(w)$ when $i > j > \alpha_1$ and $\alpha$ is join.

Summed up nicely by pattern avoidance!
Domains for type-B compositions

For the join case, exchange (?) the roles of the root and the second root.

- Highest \( y \) on \( x = 0 \): \( \alpha_1 + \# \) children of the second root.
- \( \# \uparrow \) on \( x = \alpha_1 \): \( \# \) children of the root not in region 1.

So that the highest \( y \)-coordinate on \( x = 0 \) is the max number of switch nodes.