On Whitney numbers of the first and second kind, or is it the other way around?

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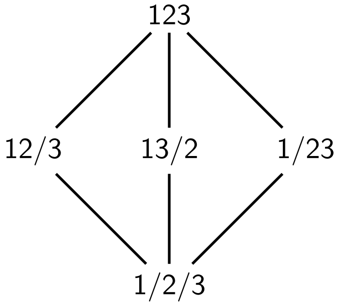
Joint work with Josh Hallam, Yeison Quiceno, Michelle Wachs.

The usual suspects

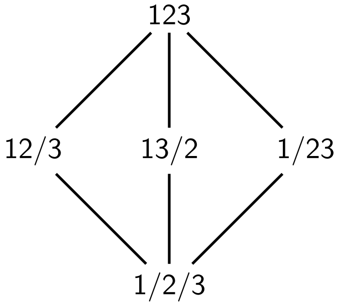
Posets!

Example of a poset



Π_3

Example of a poset

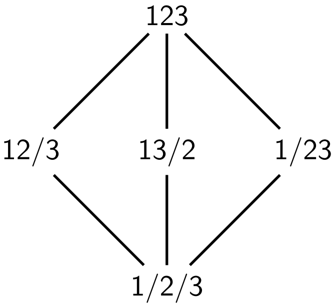


Π_3

Terminology: bottom, pure or graded

The Möbius function

The Möbius function



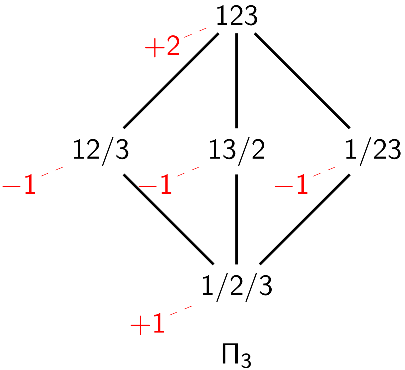
Π_3

Recursive definition:

$$\mu(\hat{0}, \hat{0}) = 1$$

$$\sum_{\hat{0} \leq z \leq x} \mu(\hat{0}, x) = 0 \text{ for } x > \hat{0}$$

The Möbius function



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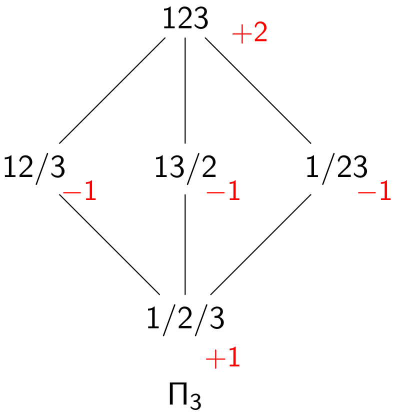
Whitney numbers

The Whitney Numbers

The k^{th}-**Whitney number of the first kind** of P is defined by

$$w_k(P) = \sum_{\rho(x)=k} \mu(\hat{0}, x)$$



$w_2(\Pi_3) = 2$

$w_1(\Pi_3) = -3$

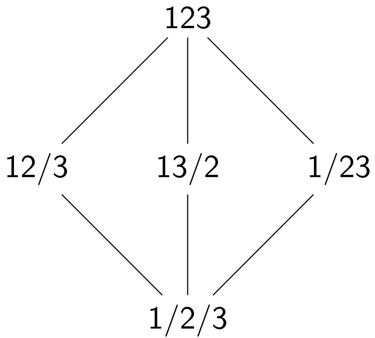
$w_0(\Pi_3) = 1$

A simpler set of Whitney numbers

The Whitney Numbers

The k^{th}-**Whitney number of the second kind** of P is defined by

$$W_k(P) = |\{x \in P \mid \rho(x) = k\}|.$$



Π_3

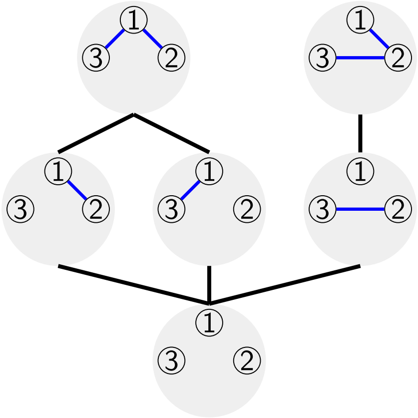
$W_2(\Pi_3) = 1$

$W_1(\Pi_3) = 3$

$W_0(\Pi_3) = 1$

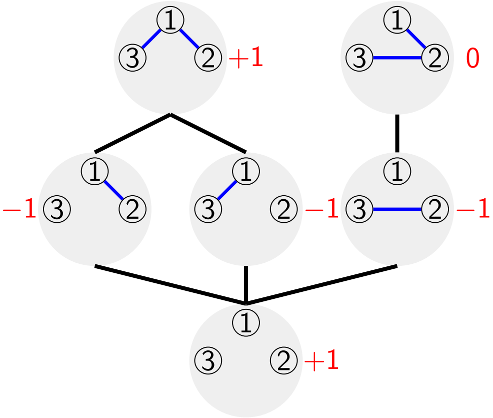
Playing this game in another poset

$\mathcal{ISF}_n$



$\mathcal{ISF}_3$

$\mathcal{ISF}_n$



$w_2(\mathcal{ISF}_3) = 1$
$W_2(\mathcal{ISF}_3) = 2$

$w_1(\mathcal{ISF}_3) = -3$
$W_1(\mathcal{ISF}_3) = 3$

$w_0(\mathcal{ISF}_3) = 1$
$W_0(\mathcal{ISF}_3) = 1$

$\mathcal{ISF}_3$

Whitney Numbers of Π_3 and $\mathcal{ISF}_3$

Compare the Whitney numbers of Π_3 and $\mathcal{ISF}_3$.

k	$w_k(\Pi_3)$	$W_k(\Pi_3)$	$w_k(\mathcal{ISF}_3)$	$W_k(\mathcal{ISF}_3)$
0	1	1	1	1
1	-3	3	-3	3
2	2	1	1	2

Whitney duals

Let P and Q be graded posets both with $\hat{0}$s. We say that P and Q are **Whitney duals** if for all k,

$$|w_k(P)| = W_k(Q) \text{ and } |w_k(Q)| = W_k(P).$$

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k	$\lvert w_k(\Pi_3)\rvert$	$W_k(\mathcal{ISF}_3)$	$\lvert w_k(\mathcal{ISF}_3)\rvert$	$W_k(\Pi_3)$
0	1	1	1	1
1	3	3	3	3
2	2	2	1	1

Thus the table above shows that Π_3 and $\mathcal{ISF}_3$ are Whitney duals. In fact, Π_n and $\mathcal{ISF}_n$ are Whitney duals for all n.

What posets have Whitney duals?

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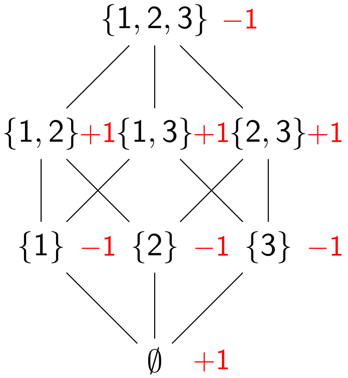
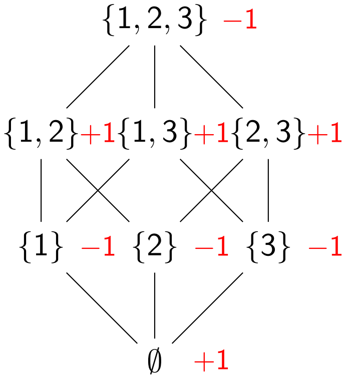


The boolean algebra is its own Whitney dual.

What posets have Whitney duals?



The boolean algebra is its own Whitney dual.

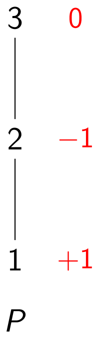
More generally, in any Eulerian poset,

$$\mu(x, y) = (-1)^{\rho(y)-\rho(x)}$$

for all x and so $|w_k(P)| = W_k(P)$ for all k. Thus, all Eulerian posets have Whitney duals, namely themselves.

What posets have Whitney duals?

There are also posets which do not have Whitney duals.



$$w_2(P) = 0$$

$$W_2(P) = 1$$

$$w_1(P) = -1$$

$$W_1(P) = 1$$

$$w_0(P) = 1$$

$$W_0(P) = 1$$

Constructing Whitney duals

Constructing Whitney duals

There are two main ingredients we use to construct Whitney duals.

1. Edge labelings
2. Quotient posets

Edge labelings and Whitney numbers

An **edge labeling** $\lambda : \mathcal{E}(P) \to \Lambda$ of a poset P, is a labeling of the set $\mathcal{E}(P)$ of edges of the Hasse diagram of P where the set of labels Λ is a partial order.

Edge labelings and Whitney numbers

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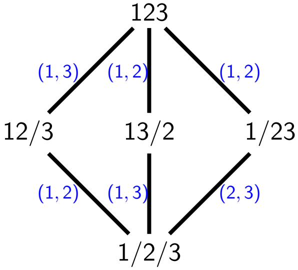
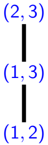


Π_3



Λ

Edge labelings and Whitney numbers

Given an edge labeling, we say a saturated chain $x_1 \lessdot x_2 \lessdot \cdots \lessdot x_k$ is **increasing** if

$$\lambda(x_1, x_2) < \lambda(x_2, x_3) < \cdots < \lambda(x_{k-1}, x_k).$$

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Similarly, we say the chain is **ascent-free** if

$$\lambda(x_1, x_2) \not< \lambda(x_2, x_3) \not< \cdots \not< \lambda(x_{k-1}, x_k).$$

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We say λ is a an **ER-labeling** if every interval has a unique increasing maximal chain.

Edge labelings and Whitney numbers

Given an edge labeling, we say a saturated chain $x_1 \lessdot x_2 \lessdot \cdots \lessdot x_k$ is **increasing** if

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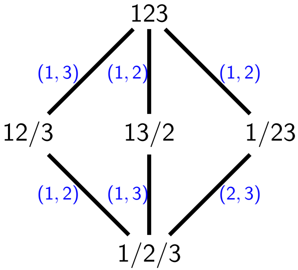
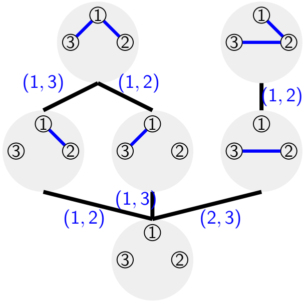
$$\lambda(x_1, x_2) \not< \lambda(x_2, x_3) \not< \cdots \not< \lambda(x_{k-1}, x_k).$$

We say λ is a an **ER-labeling** if every interval has a unique increasing maximal chain. On the other hand, we say λ is an **ER*-labeling** if every interval has a unique ascent-free maximal chain.

Edge labelings and Whitney numbers



Π_3



$\mathcal{ISF}_3$

The labeling on Π_3 is an ER-labeling (unique maximal increasing chain in each interval) and the labeling on $\mathcal{ISF}_3$ is an ER*-labeling (unique maximal ascent-free chain in each interval).

Edge labelings and Whitney numbers

Theorem (Stanley)

Let P be a graded poset with an ER-labeling. Then

$$\mu(x, y) = (-1)^{\rho(y)-\rho(x)} |\{\mathbf{c} \mid \mathbf{c} \text{ is an ascent-free maximal chain in } [x, y]\}|.$$

Edge labelings and Whitney numbers

Theorem (Stanley)

Let P be a graded poset with an ER-labeling (ER*-labeling). Then

$$\mu(x,y) = (-1)^{\rho(y)-\rho(x)} |\{\mathbf{c} \mid \mathbf{c} \text{ is an ascent-free (increasing) maximal chain in } [x,y]\}|.$$

Edge labelings and Whitney numbers

We can summarize the implications of this result with respect to Whitney numbers in the following table.

	$\lvert w_k(P)\rvert$	$W_k(P)$
λ is an ER-labeling	# (ascent-free sat. chains of length k starting at $\hat{0}$)	# (increasing sat. chains of length k starting at $\hat{0}$)
λ is an ER*-labeling	# (increasing sat. chains of length k starting at $\hat{0}$)	# (ascent-free sat. chains of length k starting at $\hat{0}$)

Whitney labelings

Whitney labelings

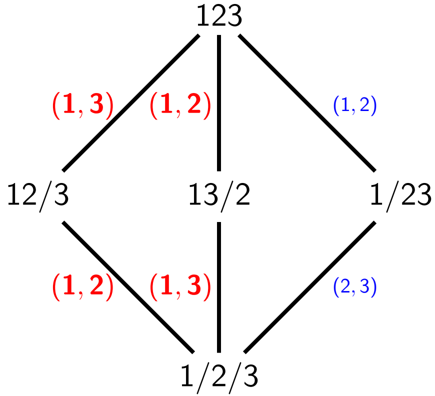
Definition

We say that an edge labeling λ is an **EW-labeling** if the following hold:

1. λ is an ER-labeling
2. λ satisfy the **rank two switching property**.
3. **An injectivity condition**

Edge labelings and Whitney numbers

Let λ be an ER-labeling of P. We say λ has the **rank two switching property** if for every interval $[x, y]$ with $\rho(y) - \rho(x) = 2$, if the unique increasing word is ab, then there is a unique maximal chain in $[x, y]$ labeled ba.



Whitney labelings and Whitney duals

Theorem (González D'León-Hallam, (2018))

Let P be a graded poset with a $\hat{0}$ and λ an EW-labeling of P. Then P has a Whitney dual.

Whitney labelings and Whitney duals

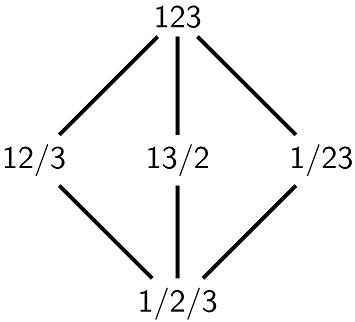
Theorem (González D'León-Hallam, (2018))

Let P be a graded poset with a $\hat{0}$ and λ an EW-labeling of P. Then P has a Whitney dual. Moreover, we can construct a particular Whitney dual $Q_\lambda(P)$ that depends on λ.

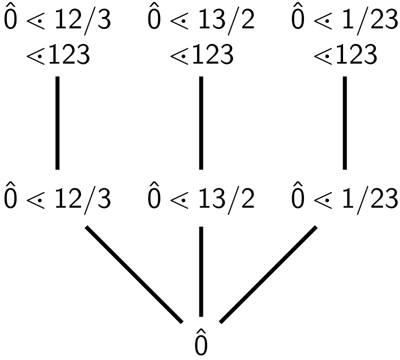
How is the construction?

Chain Posets

The **chain poset** $C(P)$ is the set of saturated chains starting at $\hat{0}$ ordered by inclusion.

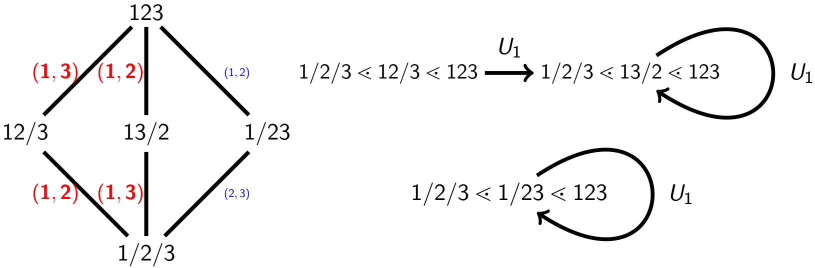


Π_3

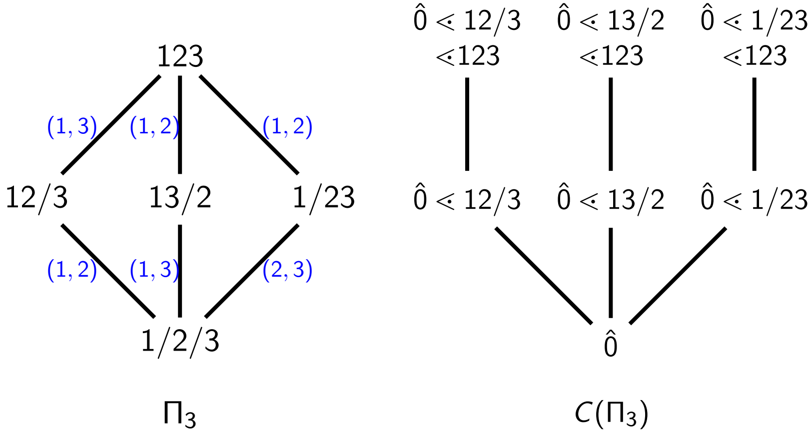


$C(\Pi_3)$

Quadratic exchange



An example of the construction



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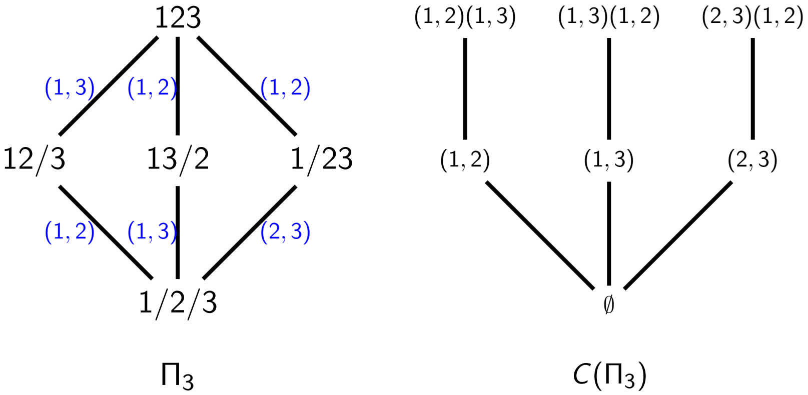

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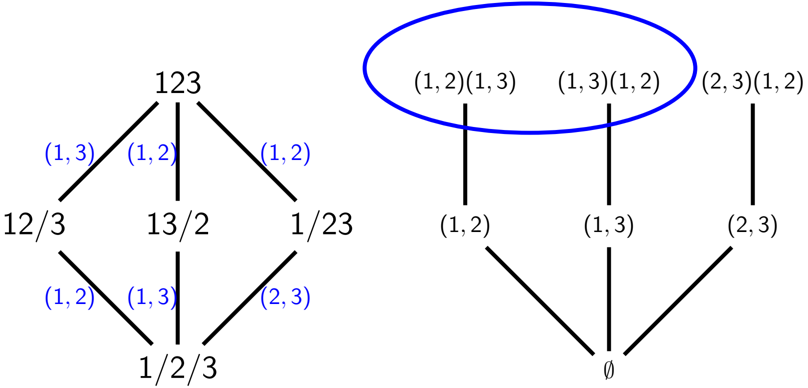
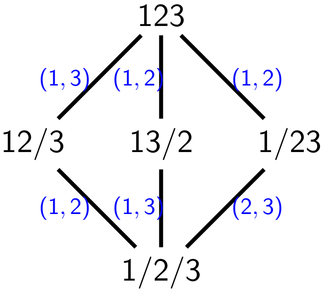
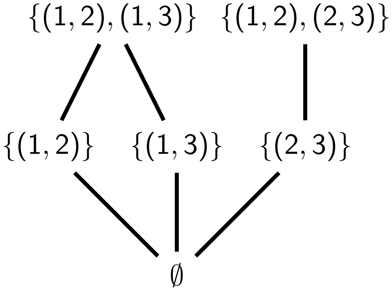


Π_3 $C(\Pi_3)$

An example of the construction

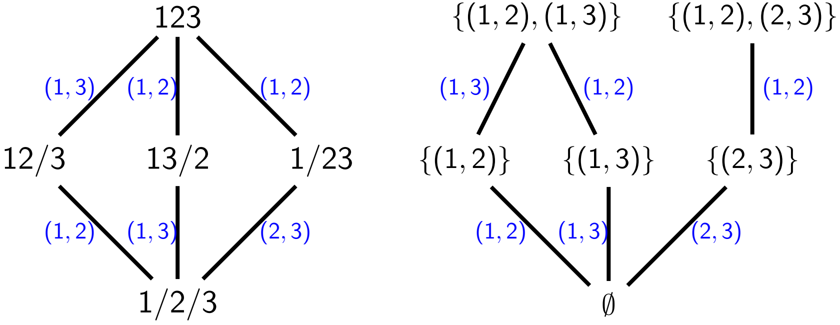


Π_3



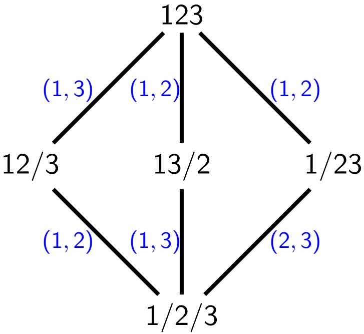
$Q_\lambda(\Pi_3)$

Labeling on $Q_\lambda(P)$

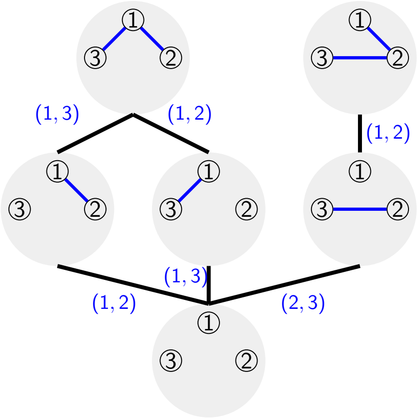


Π_3 $\qquad$ $Q_\lambda(\Pi_3)$

The dual $Q_\lambda(P)$



Π_3



$Q_\lambda(\Pi_3)$

Main result

Theorem (González D'León - Hallam (2018))

Let P be a graded poset with a $\hat{0}$ and λ an EW-labeling of P. Then P has a Whitney dual $Q_\lambda(P)$.

Examples!

Definition (Stanley)

Let L be a geometric lattice and let $\preceq$ be a total order on the atoms of L. Define $\lambda(x \lessdot y) = a$ to be the smallest (with respect to $\preceq$) atom such that $a \vee x = y$.

Geometric Lattices

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Theorem (G. D'L. - Hallam (2017))

Stanley's labeling of geometric lattices is an EW-labeling and hence every geometric lattice has a Whitney dual.

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Theorem (G. D'L. - Hallam (2017))

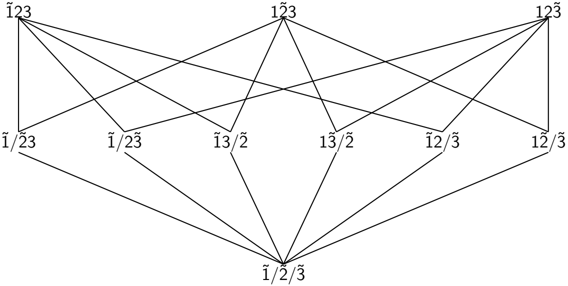
Stanley's labeling of geometric lattices is an EW-labeling and hence every geometric lattice has a Whitney dual.

Moral: Geometric lattices are Whitney dualizable.

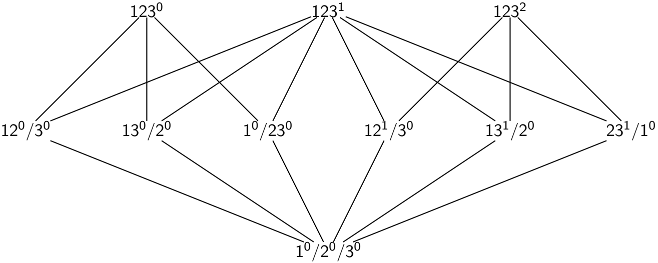
A pair of operadic examples

Posets of pointed and weighted partitions

$\Pi_n^\bullet$



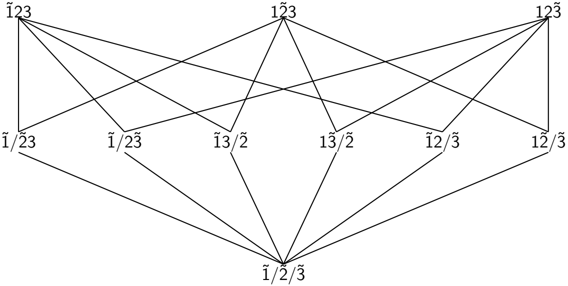
Π_n^w



Posets of pointed and weighted partitions

$\Pi_n^\bullet$

D.L. Reiner (1977)



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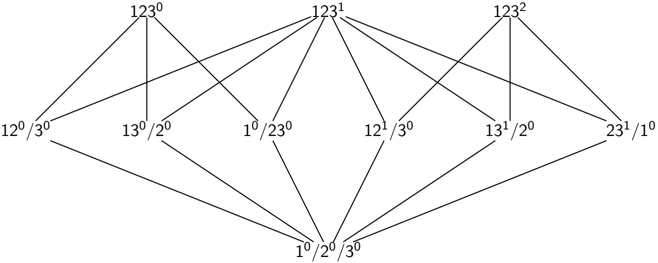

Posets of pointed and weighted partitions

$\Pi_n^\bullet$

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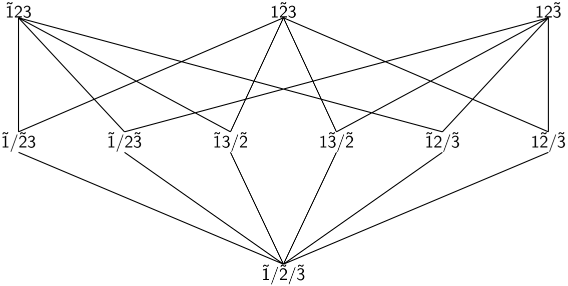
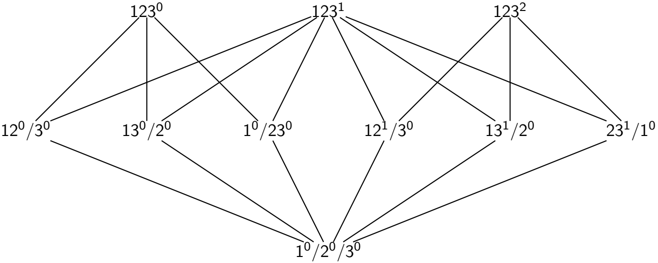
Chapoton-Vallette (2006)

Operad $\mathcal{P}erm$



Π_n^w



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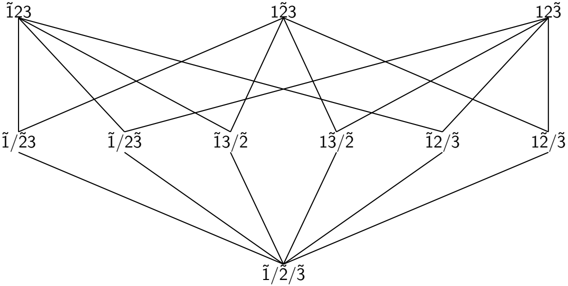
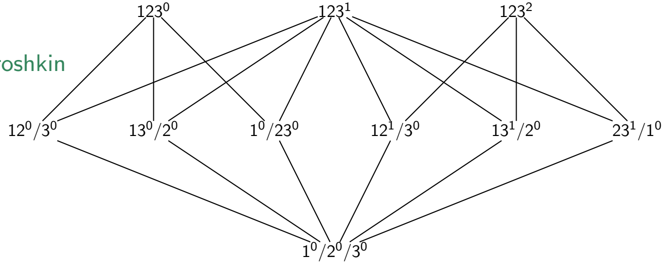


Π_n^w

Dotsenko-Khoroshkin (2007)

Operad ${}^2\mathcal{C}om$



Posets of pointed and weighted partitions

$\Pi_n^\bullet$

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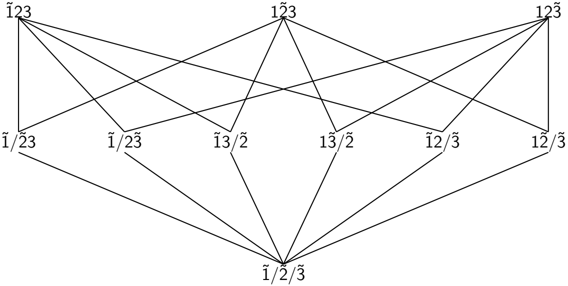

Π_n^w

Dotsenko-Khoroshkin (2007)

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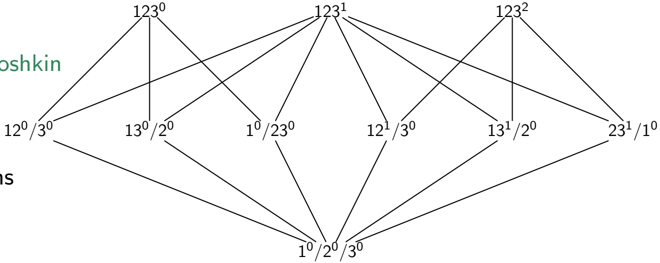
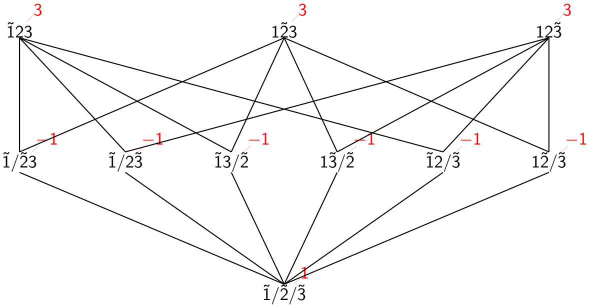
G.D'L.- Wachs (2014)

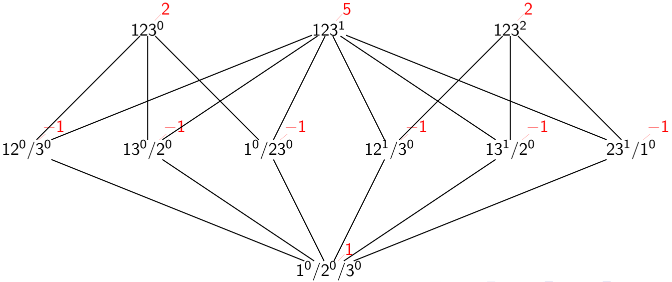


Posets of pointed and weighted partitions

$\Pi_n^\bullet$



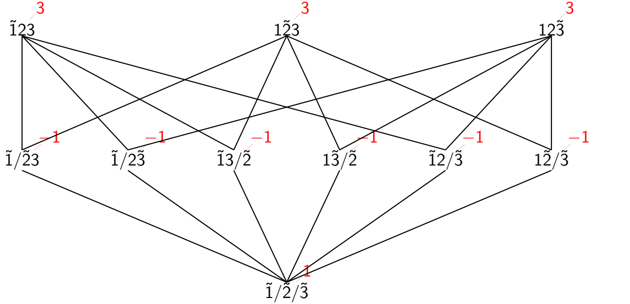
Π_n^w



Posets of pointed and weighted partitions

$\Pi_n^\bullet$

	w_k	W_k
	9	3
	-6	6
	1	1



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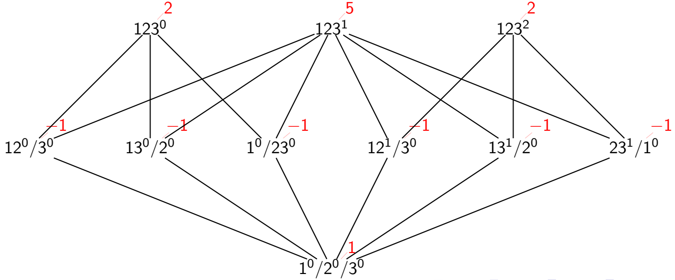


Operadic partition posets

Theorem (G. D'L. - Wachs 2014)

$\Pi_n^\bullet$ and Π_n^w have the same Whitney numbers of first and second kind.

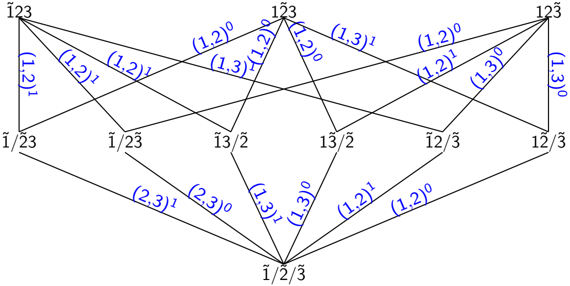
Theorem (G. D'L. - Wachs 2014)

$\Pi_n^{\bullet}$ and Π_n^{w} have the same Whitney numbers of first and second kind.

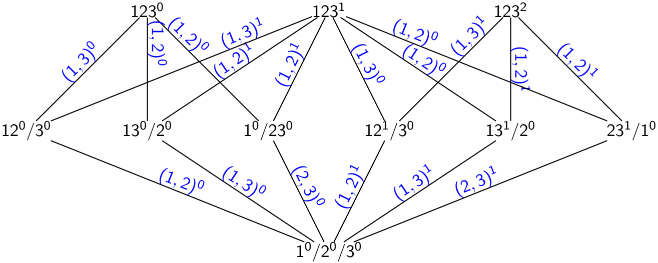
We call such posets Whitney twins

Posets of pointed and weighted partitions

$\Pi_n^\bullet$



Π_n^w



Posets of pointed and weighted partitions

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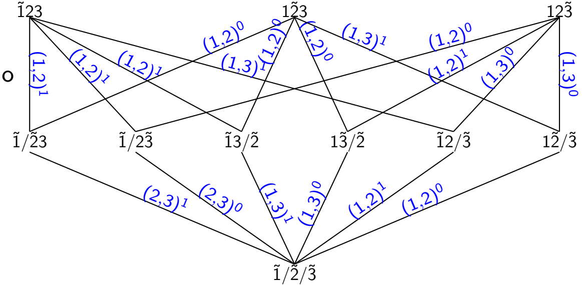
G. D'L-Hallam-Quiceno (2023): EW



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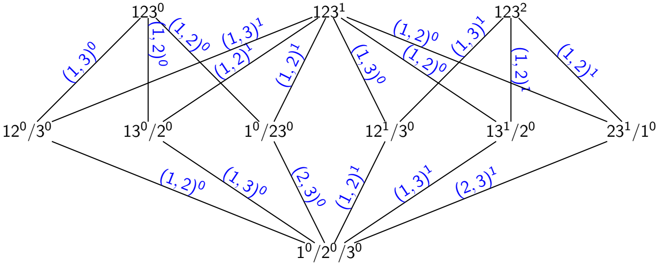

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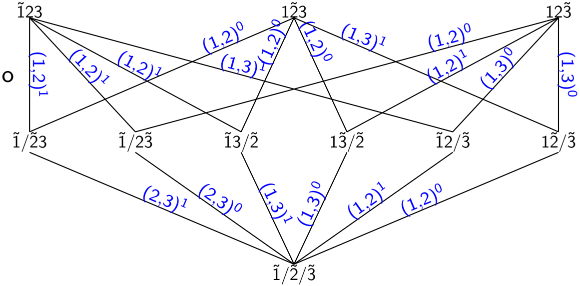
(2023): EW

(2023): dual EL



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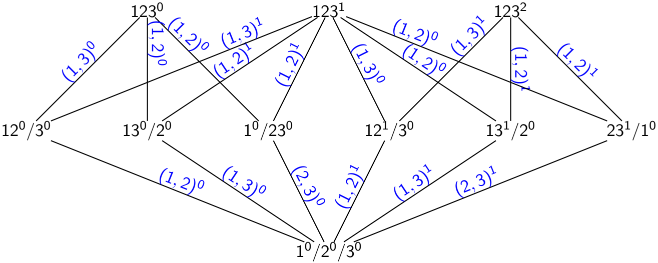

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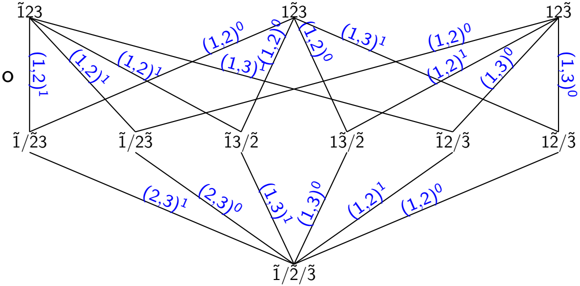

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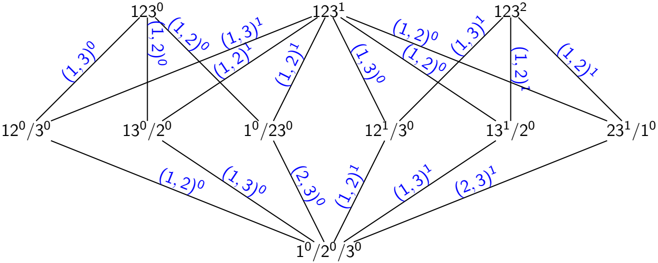

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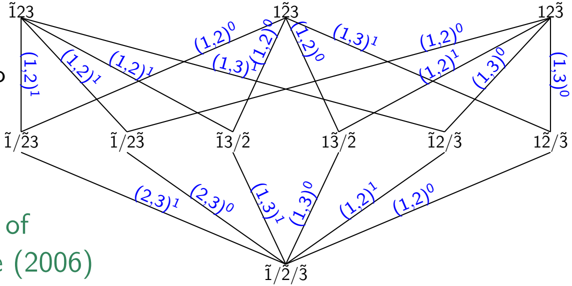
Answers question of Chapoton-Vallette (2006)



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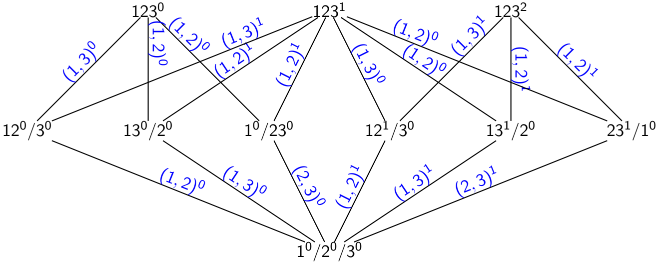

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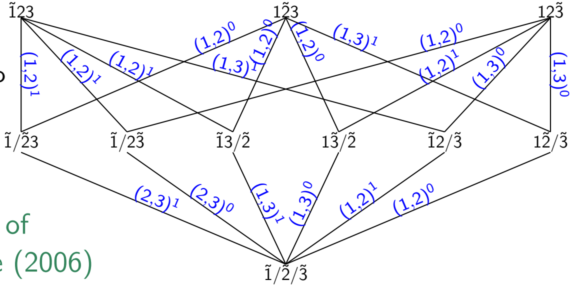
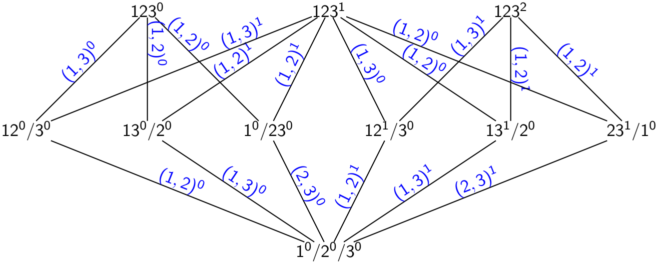


Π_n^w

G.D'L-Wachs

(2014): EL



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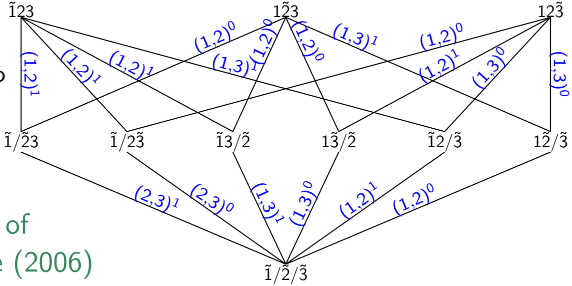

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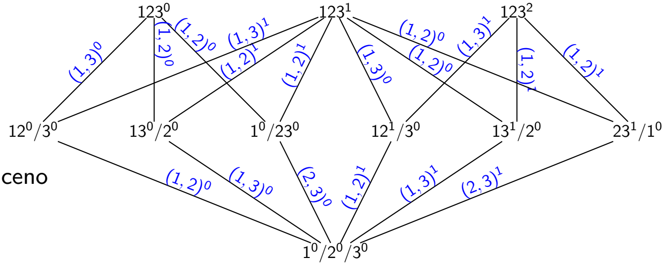

Operadic partition posets

Theorem (G. D'L. - Hallam - Quiceno (2023))

The Whitney duals of $\Pi_n^\bullet$ and Π_n^w associated to their EW-labelings are not isomorphic for $n \geq 4$.

Operadic partition posets

Theorem (G. D'L. - Hallam - Quiceno (2023))

The Whitney duals of $\Pi_n^\bullet$ and Π_n^w associated to their EW-labelings are not isomorphic for $n \geq 4$.

Moral: Whitney duals are not unique.

Operadic partition posets:Related results

Theorem (G. D'L. - Hallam - Quiceno (2023), G. D'L.- Wachs (2014))

The posets $\Pi_n^\bullet$ and Π_n^w have EL-labelings and hence their maximal intervals are Cohen-Macaulay.

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Corollary (Chapoton-Vallette (2006), Dotsenko-Khoroshkin (2010))

$\mathcal{P}erm$, $\mathcal{P}re\mathcal{L}ie$, ${}^2\mathcal{C}om$, *and* $\mathcal{L}ie^2$ *are Koszul operads.*

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Corollary (G. D'L. - Hallam - Quiceno (2023),G. D'L. - Wachs (2014))

The ascent-free chains of these EL-labelings give

- *Two bases for $\mathcal{P}re\mathcal{L}ie$,*
- *A basis for $\mathcal{L}ie^2$.*

Operadic partition posets: Related results

Theorem (González D'León - Hallam - Quiceno 2023)

The EL-labelings for $\Pi_n^\bullet$ and Π_n^w are compatible with isomorphisms according to the definition in Bellier-Millès, Delcroix-Oger, and Hoffbeck (2021).

Operadic partition posets: Related results

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Corollary (González D'León - Hallam - Quiceno 2023)

The increasing chains of the EL-labelings give PBW bases for $\mathcal{P}erm$ and ${}^2\mathcal{C}om$.

Gracias!