(q,t)-symmetry in triangular partitions

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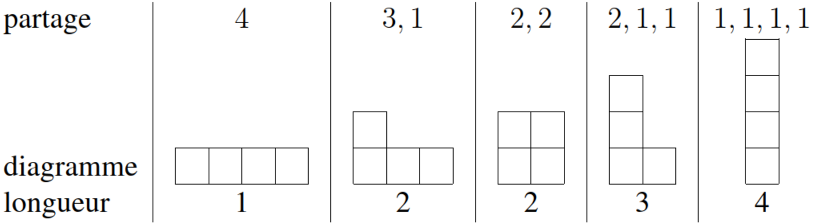
1. Preamble
2. Results on triangular partitions
3. Lattice on triangular partitions

Partition

- A partition of n is a way to represent n as a sum of positive integers (the number of zero being irrelevant)
- We write them as k-uplet $(i_1, i_2, ..., i_k)$ with $i_1 \geq i_2 \geq ... \geq i_k$
- An other representation would be with Ferrer diagrams



Triangular partitions: definition

- A partition is said to be triangular if it is the biggest partition under a line going from $(r, 0)$ to $(0, s)$ with r and s two real numbers

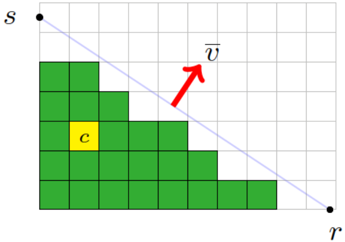


Figure 1: A triangular partition, figure made by François Bergeron

Frame of work

When we study the symmetry, we fix a partition λ and only study its subpartitions.

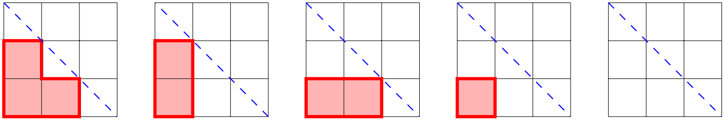


Figure 2: All the subpartitions of (2,1)

Subpartitions of a step-partition

Subpartitions of a step-partition can easily be seen as Dyck paths of the same size than than step-partition.

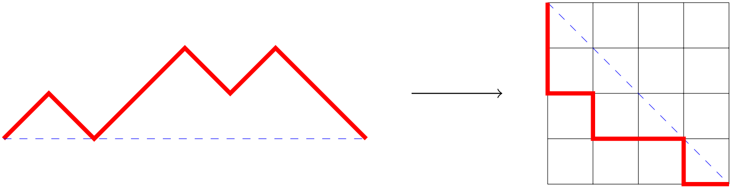


Figure 3: On the left, a Dyck path of size 3, on the right, the subpartition of the step-partition of size 3 associated

Example of a (q,t) enumeration on (3,2)

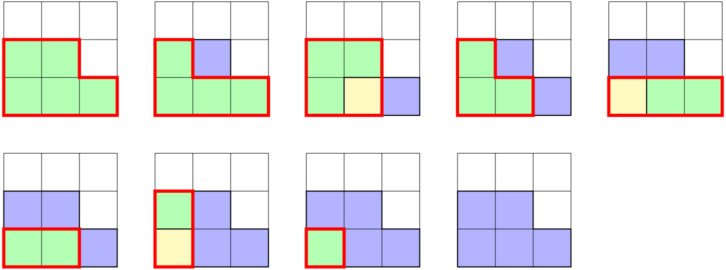


Figure 4: All subpartitions of (3,2) with, in green, the similar cells of the subpartitions, in yellow, cells of the subpartitions that aren't similar and in blue, cells that aren't in the subpartition but are in the fixed partition.

For (3,2), we will get the following polynomial
$F(q,t) = t^5 + t^4 q + t^3 q + t^3 q^2 + t^2 q^2 + t^2 q^3 + tq^3 + tq^4 + q^5$

The algebraic formula

The formula for the symmetric function of Catalan objects is the following:

$$a_\tau(q,t) = \sum_{\mu \vdash l(v_\tau)} \sum_{\theta \in \mathrm{SYT}(\mu)} \Omega_\mu(\theta) \prod_{(i,j)\mu} (q^i t^j)^{v_\tau(\theta(i,j))}.$$

with $v_\tau = (0, \tau_1 - \tau_2, \dots, \tau_i - \tau_{i+1}, \dots)$, and

$$\Omega(\theta) := \prod_{\theta(a,b) > \theta(i,j)} \frac{(q^a t^b - q^i t^j)^* (q^a t^b - q^{i+1} t^{j+1})^*}{(q^a t^b - q^{i+1} t^j)^* (q^a t^b - q^i t^{j+1})^*} *$$

$$\prod_{\theta(a,b) = \theta(i,j)+1} \frac{q^a t^b}{q^a t^b - q^{i+1} t^{j+1}} \prod_{\theta(i,j) \neq 1} \frac{1}{q^i t^j - 1},$$

with $(A)^*$ equal 1 if A is null, and A otherwise.

Similar sub-partition

Definition *Similar subpartition*:A subpartition in which each cells are similar.
For example in (3,2), the similar subpartitions were:
$(3, 2), (3, 1), (2, 1), (2), (1)and()$.
Proposition: There is exactly one such sub-partition for each size smaller than the original partition.

qt-symmetry in triangular partitions

Definition

V. PONS, L.L.M. Creation of the triangular Young tableau: a tool to easily obtained the sim

Created with the similar sub-partitions of the studied triangular partition

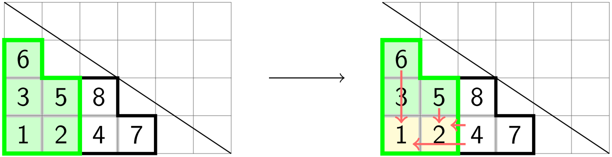


Figure 5: On the left, the partition (4,3,1) with it's triangular Young tableau and it's subpartition (2,2,1). On the right, how to get the sim of (2,2,1) with the tableau: 2 deficit cells, so a sim of 3 and an area of 3.

qt-symmetry in triangular partitions

Theorem

For a triangular 2-partition $\tau = mn$, the associated symmetric function is :

$$a_\tau(q,t) = \sum_{i=0}^{n} s_{(m+n-2i,i)}$$

- It's a result first algebraically proven by François Bergeron a few months before
- The proof being purely combinatorial, it gives use a combinatorial proof of the (q,t)-symmetry in 2-partitions.
- it is, as such, an answer to one of the open questions, in an infinity of particular cases.

The sim-sym tableaux

Theorem

Characterisation of standard Young tableau giving the expected symmetric function when we compute the sim using said tableau. This c=tableaux are also called sim-sym tableaux.

We observe the need to alternate between the two parts of the 2-partitions

6	8				
1	2	3	4	5	7

5	7				
1	2	3	4	6	8

4	6				
1	2	3	5	7	8

3	5				
1	2	4	6	7	8

2	5				
1	3	4	6	7	8

Figure 6: The 5 different symmetric tableau on (6,2).

Lattices on triangular partitions

The previous symmetric functions can be extended on 3 variables.

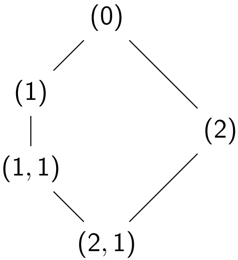


Figure 7: Tamarri lattice on sub-partition of (2,1)

For 3 variables, $F_3(q,t,r) = s_3(q,t,r) + s_{1,1}(q,t,r)$. Thus $F_3(1,1,1) = 13$, which is also the number of intervals of the lattice in the figure 7.

Lattice on 2-partitions

- A generalization of Tamari lattices, the ν-Tamari lattices, given by Préville-Ratelle and Viennot
- They don't work on all triangular partitions (not always the expected number of intervals, for example), but work on 2-partitions

Cover relation in ν-Tamari

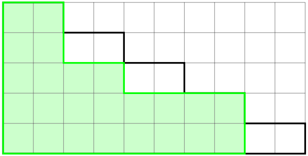
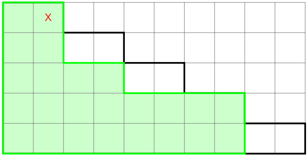


Figure 8: Covers relation of the subpartition (8,8,4,2,2).

Cover relation in ν-Tamari



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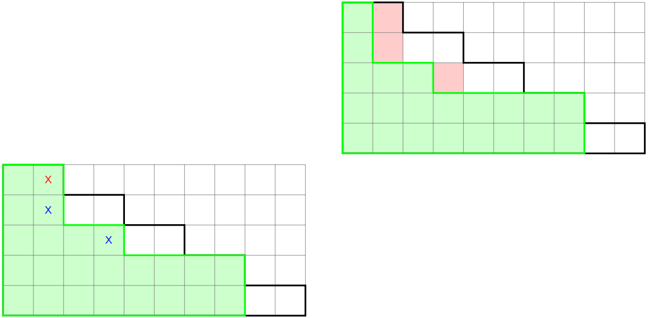

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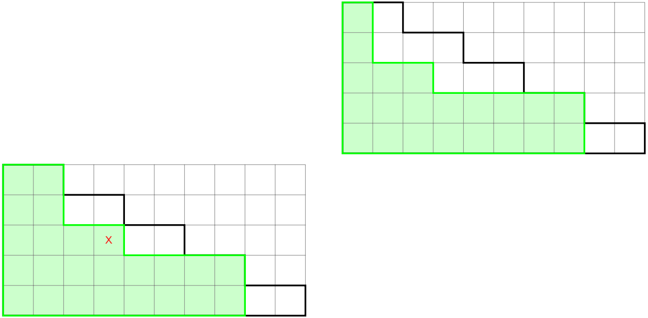

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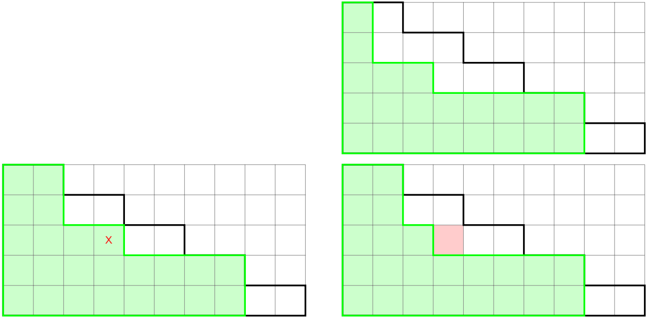

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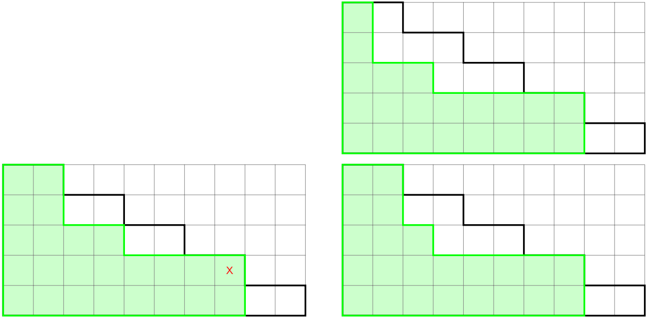

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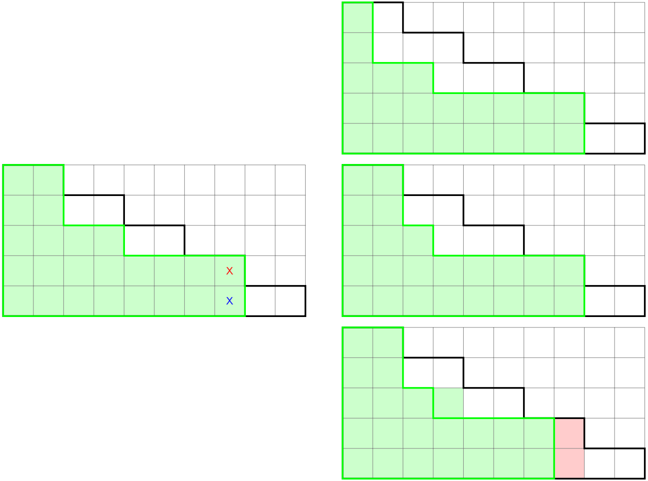

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Exemple of a ν-Tamari lattice on 2-partition

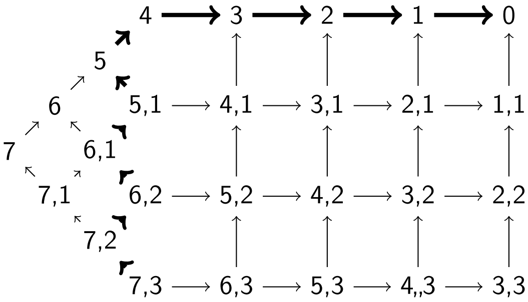

Figure 9: ν-Tamari for (7,3).

We define the *topdown* tableau with the chain of maximal length. This tableau is indeed sim-sym and brings a better understanding of the (q, t, r)-symmetry in the lattice.

Lattice on a 2-partition

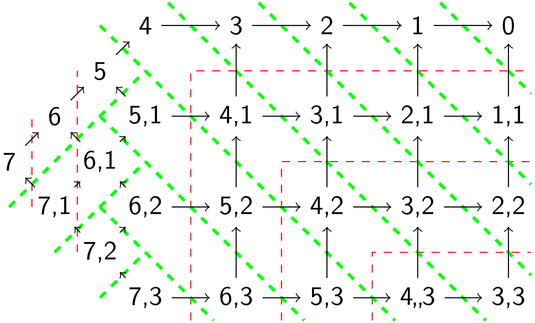

Figure 10: The ν-Tamari lattice of (7,3).

Symmetry in the lattice of 2-partitions

Theorem

On a fixed partition $\lambda = (m, n)$, the distance and the simmax are symmetric on the lattice of λ. More precisely, if we consider $R(\lambda)$ the set of relations in the lattice of λ, we have:

$$\sum_{(\alpha,\beta)\in R(\lambda)} q^{dist(\alpha,\beta)} t^{sim(\beta)} = \sum_{k=0}^{min(n,m-n)} s_{(m+n-2k,k)}(q, t, 1)$$

Merci pour votre attention!