# Représentations de $S_n$ & Mots de Smirnov

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# THE FROBENIUS CHARACTERISTIC

symmetric functions $/\mathbb{C}$

- $V$ a finite-dimensional module for $S_n / \mathbb{C}$

$\rightsquigarrow$ Frob : $\bigoplus_n \mathrm{mod}(S_n) \xrightarrow{\sim} \mathrm{Sym}_{\mathbb{C}}$ $\quad (\mathrm{Frob}(V_\lambda) = s_\lambda)$

Frob$(V)$ is Schur-positive $\rightsquigarrow$ $\mathrm{Frob}(V) = \sum_{\mathrm{obj}} x^{\mathrm{wt}(\mathrm{obj})}$

COMBINATORICS

- If $V$ has grading $V = \bigoplus_i V_i$ $\quad (S_n \circlearrowright V_i)$

$$\mathrm{Frob}_q(V) := \sum_i q^i\, \mathrm{Frob}(V_i) \in \mathrm{Sym}_{\mathbb{C}(q)}$$

Extend to.. $V = \bigoplus_{ijk\ldots} V_{ijk\ldots} \implies \mathrm{Frob}_{q_1, q_2, q_3, \ldots}(V)$

# COINVARIANT MODULE

$\mathbb{C}[x_1, x_2, \dots, x_n] = \mathbb{C}[\overline{x}]$ $\quad \sigma . x_1^{a_1} \cdots x_n^{a_n} = x_{\sigma(1)}^{a_1} \cdots x_{\sigma(n)}^{a_n}$

Invariants $= \{ P \,/\, \sigma . P = P \} = \mathbb{C}[\overline{x}]^{S_n}$

$C_{1,0}(n)$ Coinvariants $= \mathbb{C}[\overline{x}] \big/ \left( \mathbb{C}[\overline{x}]^{S_n} \right)^{+}$ $\leftarrow$ ideal $\langle P \in \mathbb{C}[x]^{S_n}$ s.t $P(0) = 0 \rangle$

graded by total degree $\rightsquigarrow$ variable $q$

FACTS:

+ $C_{1,0}(n)$ has dimension $n!$
+ Isomorphic to regular representation $\bigoplus_{\lambda \vdash n} f_\lambda V_\lambda$
+ $\mathrm{Frob}_q(V)$ well-understood combinatorially

# DIAGONAL ACTION

$$S_n \curvearrowright \mathbb{C}[\underbrace{\bar{x}}_{(x_1,\dots,x_n)}, \underbrace{\bar{y}}_{(y_1,\dots,y_n)}, \bar{z}, \dots] \qquad \sigma \cdot y_i = y_{\sigma(i)},\ \sigma \cdot z_i = z_{\sigma(i)}, \dots$$

$$C_{r,0} := \frac{\mathbb{C}[\bar{x}^{(1)}, \dots, \bar{x}^{(r)}]}{\left(\mathbb{C}[\bar{x}^{(1)}, \dots, \bar{x}^{(r)}]^{S_n}\right)^+}$$

<u>$r=2$</u>: $C_{2,0}(n)$ = usual diagonal coinvariants $= \mathbb{C}[\bar{x},\bar{y}] / \mathbb{C}(\bar{x},\bar{y})^{S_n +}$

$\dim C_{2,0}(n) = (n+1)^{n-1}$, $\mathrm{Frob}_{q,t}$ known ... DIFFICULT

<u>$r=3$</u> $\dim C_{3,0}(n) \overset{\text{CONJ}}{=}$ # of labeled Tamari intervals

$r > 3$ ?

# EVEN MORE VARIABLES

More recently, anticommutative variables were considered

$\theta_1, \dots, \theta_n$ satisfy $\theta_i^2 = 0$, $\theta_i\theta_j = -\theta_j\theta_i$ $i,j = 1,\dots,n$

$\sigma \cdot \theta_i = \theta_{\sigma(i)}$ $\sigma \in S_n$

$$\boxed{C_{r,s}(n) := \mathbb{C}[\overline{x}^{(1)},\dots,\overline{x}^{(r)},\overline{\theta}^{(1)},\dots,\overline{\theta}^{(s)}] \Big/ \left(\mathbb{C}[\overline{x}^{(1)},\dots,\overline{x}^{(r)},\overline{\theta}^{(1)},\dots,\overline{\theta}^{(s)}]^{S_n}\right)^+}$$

Generalized coinvariant module

# DIMENSIONS OF THE MODULES $C_{r,s}(n)$

| | $r \backslash s$ | $0$ | $\bar{\theta}$<br>$1$ | $\bar{\theta}, \bar{\xi}$<br>$2$ |
|---|---|---|---|---|
| | $0$ | $1$ | $2^{n-1}$ | $\binom{2n-1}{n}$ |
| $\bar{x}$ | $1$ | $n!$ | $\sum_{i=1}^{n} i! \left\{ {n \atop i} \right\}$ | $2^{n-1}\, n!$ |
| $\bar{x}, \bar{y}$ | $2$ | $(n+1)^{n-1}$ | $\sum_{i=0}^{n+1} \binom{n+1}{i} \frac{i^n}{2(n+1)}$ | ? |
| $\bar{x}, \bar{y}, \bar{z}$ | $3$ | $2^n (n+1)^{n-2}$ | ? | ? |

(F. Bergeron)

# OUR CONTRIBUTION

[D'Addario, Iraci, VandenWyngaerd] define a symmetric function $\Phi_{q;u,v}^{(n)}$ with coeffs polynom in $q,u,v$

$$\boxed{\text{CONJECTURE: } \quad \mathrm{Frob}_{q;u,v}(C_{2,1}(n)) = \Phi_{q;u,v}(n)}$$

Definition: A segmented Smirnov word $w$ of length $n$ is a sequence of words $(w^{(1)} \mid w^{(2)} \mid \dots \mid w^{(m)})$, $w^{(i)} \neq \varepsilon$,

- $w^{(i)}$ is a Smirnov word: it has letters in $\mathbb{Z}_{>0}$ and adjacent letters are distinct
- $n = \sum_{i=1}^{m} |w^{(i)}|$

**THEOREM** (Iraci–N.–Vanden Wyngaerd '23+)

$$\phi_{q;u,v} = \sum_{w \in SSW(n)} q^{ssinv(w)} u^{des(w)} v^{asc(w)} x_w$$

Example:

$$\begin{array}{cccccccccccc} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 & w_9 & w_{10} & w_{11} & w_{12} \\ \end{array}$$

$$\underbrace{2\,1\,2}_{w^{(1)}} \Big| \underbrace{3\,4\,1\,2\,4\,2}_{w^{(2)}} \Big| \underbrace{1\,2\,5}_{w^{(3)}} \qquad x_1^3 x_2^5 x_3 x_4^2 x_6 u^5 v^3 q$$

$x_w$, | $asc(w)$ | $des(w)$ | $ssinv(w)$

$x_w \to x_{w_1} x_{w_2} \cdots x_{w_n}$

$asc(w) \to w_i < w_{i+1}$ same segment

$des(w) \to w_i > w_{i+1}$ same segment

$ssinv(w) \to \#\{ i < j$ s.t. $w_i > w_j$

and (i) $j = 1^{st}$ posit° of segment

or (ii) $w_{j-1} > w_j$

or (iii), (iv), ... $\}$

PREUVE: Both sides satisfy the same recursion

$$\mathsf{SW}_q(\mu, k-r, \ell-a) = \sum_{r=0}^{j}\sum_{a=0}^{j}\sum_{i=0}^{j} \begin{bmatrix} n-k-\ell \\ j-r-a+i \end{bmatrix}_q \begin{bmatrix} n-k-\ell-(j-r-a+i)-1 \\ i \end{bmatrix}_q \times q^{\binom{r-i}{2}} \begin{bmatrix} n-k-\ell-(j-r-a+i) \\ r-i \end{bmatrix}_q \times q^{\binom{a-i}{2}} \begin{bmatrix} n-k-\ell-(j-r-a+i) \\ a-i \end{bmatrix}_q \mathsf{SW}_q(\mu^-, k-r, \ell-a)$$

Here $\mu = (\underbrace{\mu_1, \ldots, \mu_k}_{\mu^-}, j)$ is a composition

$SW_q(\alpha; i, j) =$ coeff of $x^\alpha u^i v^j$ in the equation

# SPECIAL CASES

- $u = v = q = 1$, coeff of $x_1 \cdots x_n$ is the number of "standard" segmented words, given by $2^{n-1} \cdot n!$

- $u = 0$ (or $v = 0$)

  $\longrightarrow$ retrieve Zabrocki's conj. for $\mathrm{Frob}_{q,v}(C_{1;1}(n))$

  (Combinatorics = ordered set partitions)

- $q = 0$ $\rightsquigarrow$ Combinatorial expansion for $\mathrm{Frob}_{u,v}((C_{0;2}(n))$

- max degree in $\{u, v\}$ $\iff$ Only one segment

  $\iff$ Smirnov words

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