Geometric realizations of the s-permutahedron

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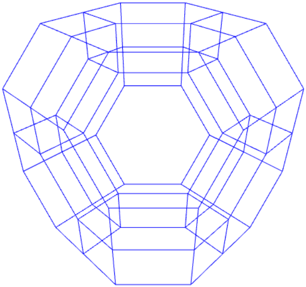
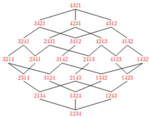
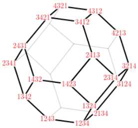
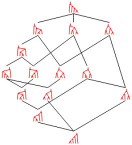
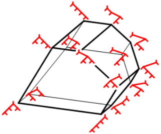
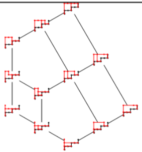
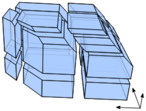


Joint work with Rafael S. González D'León, Alejandro H. Morales, Daniel Tamayo Jiménez, and Martha Yip

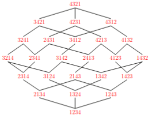
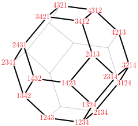
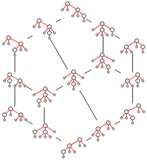
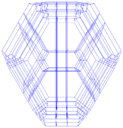
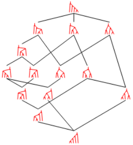
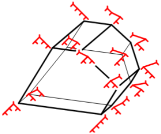
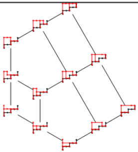
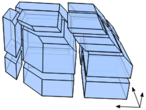
Outline

1. The s-weak order and the s-permutahedron
2. Triangulation of a flow polytope
3. Mixed subdivision of a sum of hypercubes
4. Polytopal subdivision delimited by an arrangement of tropical hypersurfaces

Motivation

Weak order	Permutahedron	?	?
			
			
Tamari lattice	Associahedron	ν-Tamari	ν-Associahedron
		Préville-Ratelle, Viennot	Ceballos, Padrol, Sarmiento

Credit: Pons '19

Weak order	Permutahedron	*s*-Weak order	*s*-Permutahedron
			
			
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s-decreasing trees (Ceballos-Pons '20)

Let $s = (s_1, \dots, s_n)$ be a (weak) composition (i.e. $s_i \in \mathbb{N}_{>0}$ or in $\mathbb{N}$).

An *s-decreasing tree* is a planar rooted tree on n internal vertices (called nodes), labeled on $[n]$ such that the node labeled i has $s_i + 1$ children and any descendant j of i satisfies $j < i$.

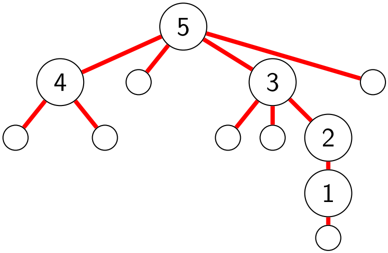


Figure: An $(0, 0, 2, 1, 3)$-decrasing tree.

s-decreasing trees and Stirling s-permutations (Ceballos-Pons '19)

Let $s = (s_1, \ldots, s_n)$ be a composition (i.e. $s_i \in \mathbb{N}_{>0}$).

An s-decreasing tree is associated to a multipermutation of $1^{s_1} \ldots n^{s_n}$ that avoids the pattern 121. Such multipermutations are called *Stirling s-permutations*.

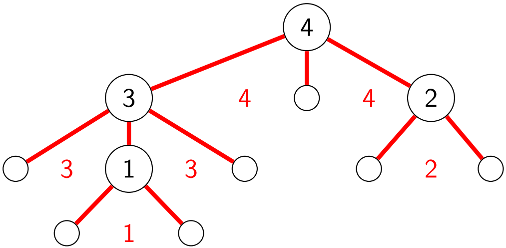
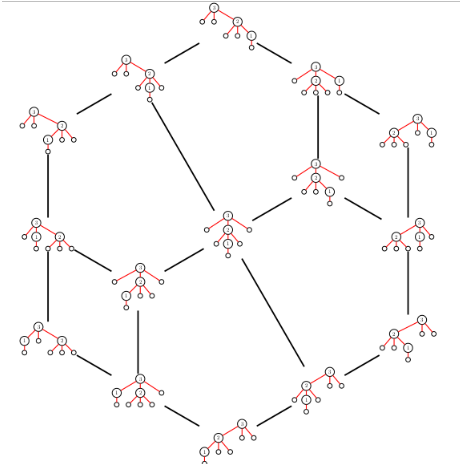


Figure: An $(1, 1, 2, 2)$-decreasing tree and the corresponding Stirling s-permutation 313442.

The s-weak order



Credit: Ceballos-Pons '19

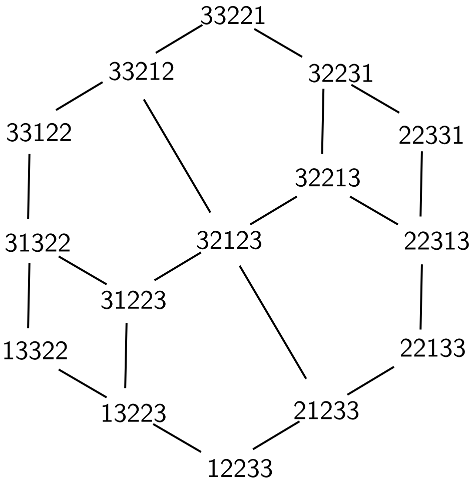


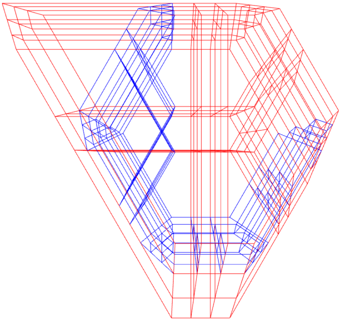
Figure: The $(1, 2, 2)$-weak order.

Conjecture 1 (Ceballos-Pons '19)

The s-permutahedron can be realized as a polyhedral subdivision of a polytope which is combinatorially isomorphic to a permutahedron.

Conjecture 2 (Ceballos-Pons '19)

If s has no zeros, there exists a geometric realization of the s-permutahedron such that the s-associahedron can be obtained from it by removing certain facets.



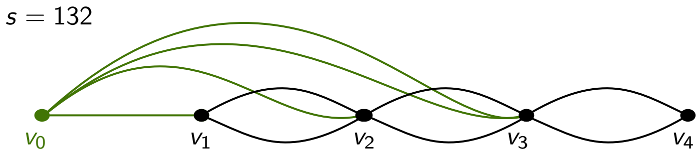
Credit: Ceballos-Pons '19

Outline

s-oruga graph

Associated to a composition $s = (s_1, \ldots, s_n)$ we consider the graph G_s on vertices $v_0, v_1, \ldots, v_{n+1}$ with:

- two edges from v_i to v_{i+1} for $i \in [n]$ and one edge from v_0 to v_1,
- $s_{n+1-i} - 1$ edges from v_0 to v_i for $i \in [n]$,
- the *framing* given by ordering incoming and outgoing edges from top top bottom on the drawing.



A *route* is a path from v_0 to v_{n+1}.

The *flow polytope* $\mathcal{F}_{G_s} = \left\{ (f_e)_{e \in E} \text{ flow of } G \right\} \subset \mathbb{R}^E$ is the convex hull of the indicator vectors of the routes of G_s.

DKK triangulation

We say that two routes P, Q of G are *coherent* with respect to the framing if they "do not cross".
For $C \in \mathcal{C}^{\max}$ (set of maximal cliques of coherent routes), Δ_C denotes the simplex with vertices the indicator vectors of the routes in C.

Theorem (Danilov-Karzanov-Koshevoy, '12)

The simplices $\{\Delta_C \mid C \in \mathcal{C}^{\max}(G, \preceq)\}$ form a (regular) triangulation of $\mathcal{F}_G$, called the DKK triangulation of $\mathcal{F}_G$ with respect to the framing $\preceq$.

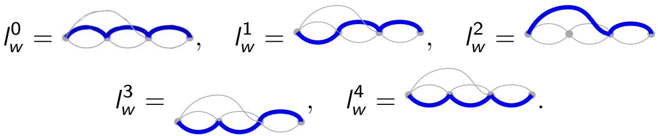


Figure: The maximal clique $\{I_w^0, \ldots, I_w^4\}$ corresponding to the $(1,2,1)$-Stirling permutation $w = 3221$.

Theorem (GMPTY, '22)

The s-decreasing trees are in bijection with the simplices of the DKK triangulation of $(\mathcal{F}_{G_s}, \preceq)$.
Moreover, two simplices are adjacent if and only if there is a cover relation in the s-weak order for the corresponding s-decreasing trees.

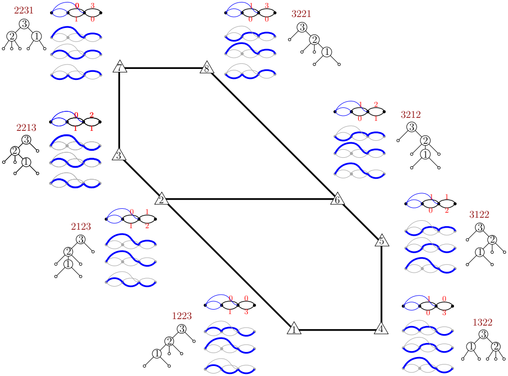
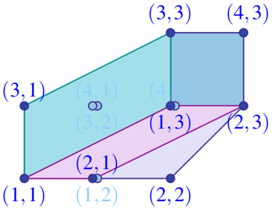


Figure: Dual of the DKK triangulation for $s = (1, 2, 1)$.

Outline

Minkowski sums

- Given polytopes $P_1, \ldots, P_k$ in $\mathbb{R}^n$, their *Minkowski sum* is the polytope $P_1 + \ldots + P_k := \{\sum x_i \mid x_i \in P_i\}$.
- The *Minkowski cells* of the sum are $\sum B_i$ where B_i is the convex hull of a subset of vertices of P_i.
- A *mixed subdivision* of a Minkowski sum is a collection of Minkowski cells such that their union covers the Minkowski sum and they intersect properly.
- A *fine mixed subdivision* is a minimal mixed subdivision via containment.



Credit: De Loera-Rambau-Santos '19

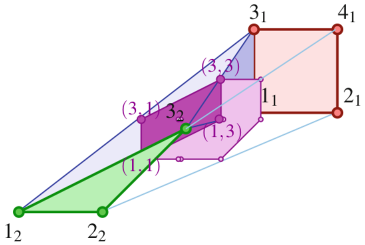
Figure: A (non fine) mixed subdivision of a sum of a square and a triangle.

Cayley Trick

$\mathcal{C}(P_1, \ldots, P_k) := \text{conv}(\{e_1\} \times P_1, \ldots, \{e_k\} \times P_k) \subset \mathbb{R}^k \times \mathbb{R}^n$ is the *Cayley embedding* of $P_1, \ldots, P_k$.

Proposition (The Cayley trick)

The (regular) polytopal subdivisions (resp. triangulations) of $\mathcal{C}(P_1, \ldots, P_k)$ are in bijection with the (coherent) mixed subdivisions (resp. fine mixed subdivisions) of $P_1 + \ldots + P_k$.



Credit: De Loera-Rambau-Santos '19

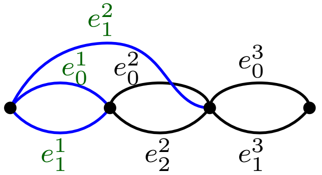
Flow polytopes are Cayley embeddings

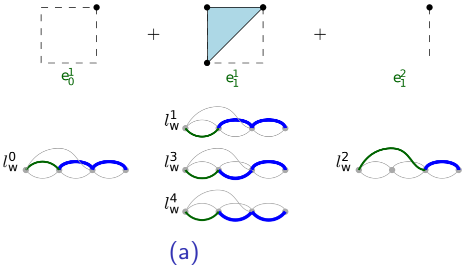
Theorem (GMPTY, '22)

The s-decreasing trees are in bijection with the maximal cells of a fine mixed subdivision of the Minkowski sum of hypercubes in $\mathbb{R}^{n-1}$ *given by*

$$(s_n + 1)\square_{n-1} + \sum_{i=1}^{n-1} (s_i - 1)\square_{i-1}.$$

Proof : The flow polytope of G_s is a Cayley embedding of hypercubes.





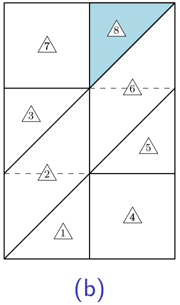
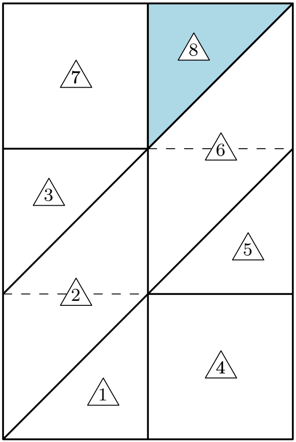
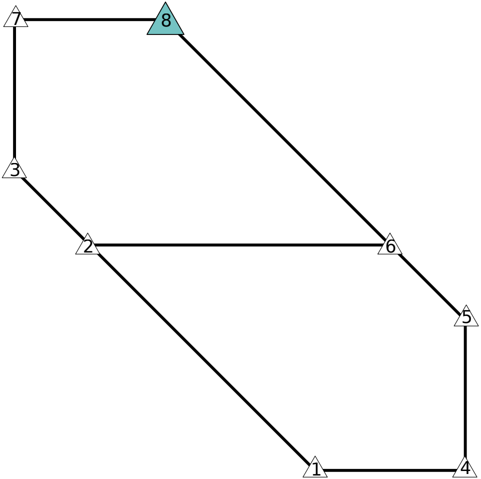


Figure: (a) Summands of the Minkowski cell corresponding to $w = 3221$.
(b) Mixed subdivision of $2\square_2 + \square_1$ realizing the $(1, 2, 1)$-permutahedron.

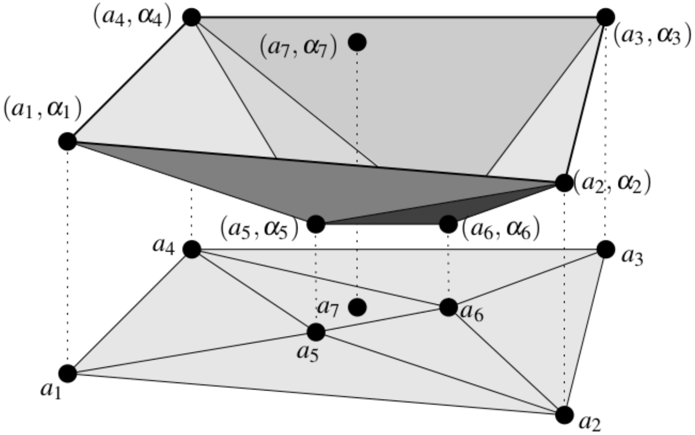
Outline

From the mixed subdivision to a dual polyhedral complex





The regular subdivision $\mathcal{S}$ of a point configuration $\mathcal{A} \subset \mathbb{R}^n$ can be obtained as the lower faces of the points of $\mathcal{A}$ lifted by an *admissible height function* α.



Credit: Rambau '96

Danilov-Karzanov-Koshevoy give an explicit admissible height function for DKK triangulations.

Such a lifted configuration is associated to a *tropical polynomial*:

$$F(x) = \bigoplus_{i \in [m]} \alpha^i \odot x^{a^i} = \min \left\{ \alpha^i + \langle a^i, x \rangle \mid i \in [m] \right\},$$

that defines the *tropical hypersurface*:

$$\mathcal{T}(F) := \{ x \in \mathbb{R}^n \mid \text{the minimum of } F(x) \text{ is attained at least twice} \}.$$

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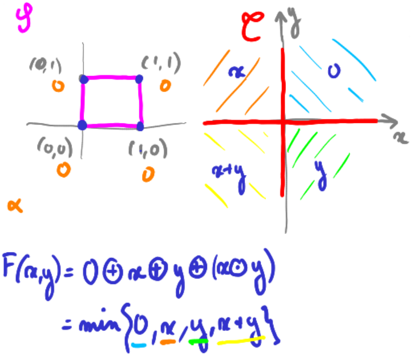

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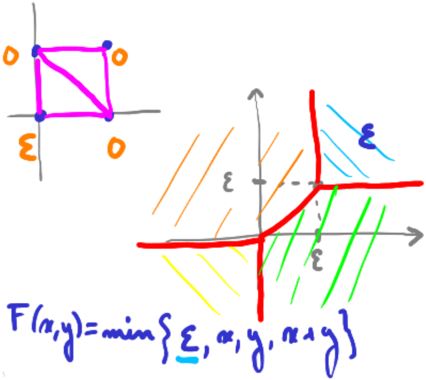

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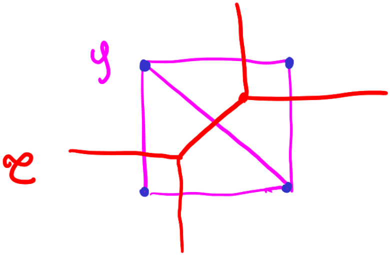
Theorem (folklore)

There is a bijection between the k-dimensional cells of $\mathcal{S}$ and the $(n-k)$-dimensional cells of $\mathcal{T}(F)$.
The bounded cells of $\mathcal{T}(F)$ corresponds to the interior cells of $\mathcal{S}$.

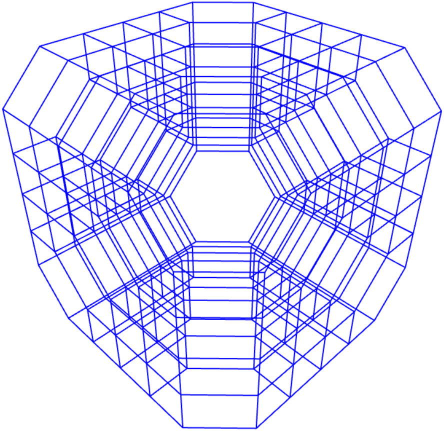


When the point configuration is a Cayley embedding, there is a factorization of the tropical polynomial of the mixed subdivision corresponding to $\mathcal{S}$ via the Cayley trick and we obtain an arrangement of tropical hypersurfaces.

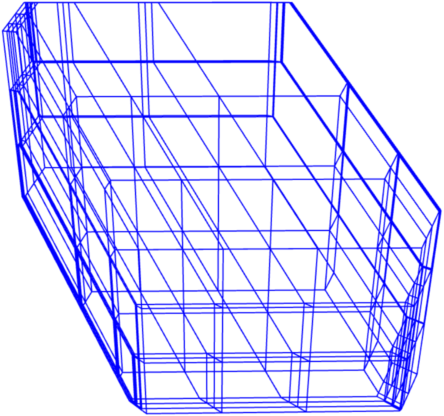
Theorem (GMPTY, '22)

The s-permutahedron can be realized as the bounded cells of an arrangement of tropical hypersurfaces.

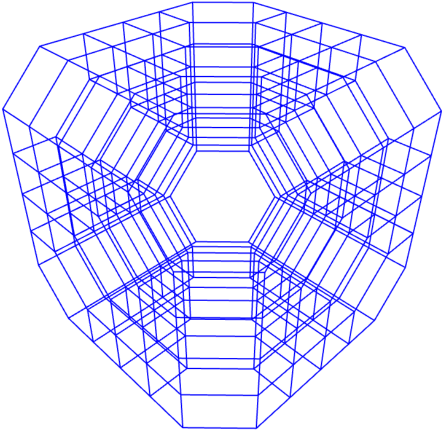
We have explicit coordinates for the vertices and all the faces!



$s = 1114$



$s = 1333$



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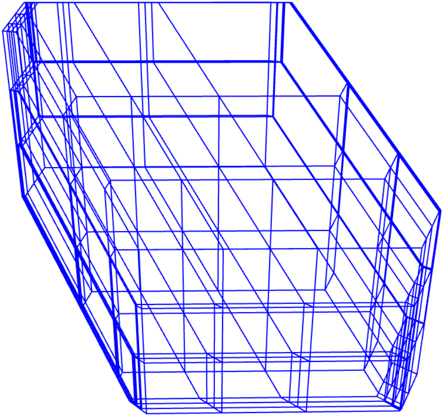

$s = 1333$

Thank you for your attention !

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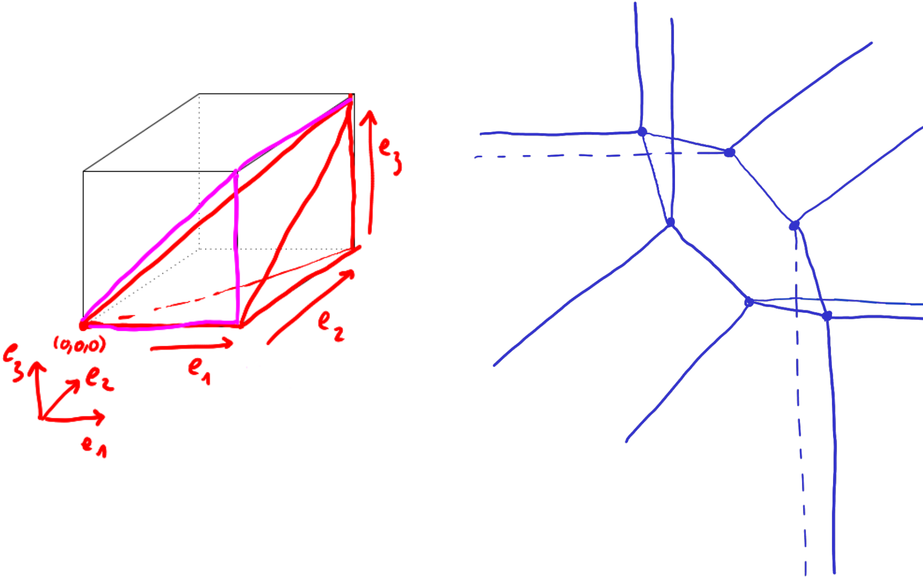
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