Geometric realizations of the s-permutahedron

Eva Philippe

Sorbonne Université

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Joint work with Rafael S. González D’León, Alejandro H. Morales, Daniel Tamayo Jiménez, and Martha Yip
1. The $s$-weak order and the $s$-permutahedron
2. Triangulation of a flow polytope
3. Mixed subdivision of a sum of hypercubes
4. Polytopal subdivision delimited by an arrangement of tropical hypersurfaces
Motivation

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**Tamari lattice**

**Associahedron**

\( \nu \)-Tamari

Préville-Ratelle, Viennot

\( \nu \)-Associahedron

Ceballos, Padrol, Sarmiento

Credit: Pons '19
## Motivation

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Credit: Pons '19

Préville-Ratelle, Viennot

Ceballos, Padrol, Sarmiento
$s$-decreasing trees (Ceballos-Pons ’20)

Let $s = (s_1, \ldots, s_n)$ be a (weak) composition (i.e. $s_i \in \mathbb{N}_{>0}$ or in $\mathbb{N}$).

An $s$-decreasing tree is a planar rooted tree on $n$ internal vertices (called nodes), labeled on $[n]$ such that the node labeled $i$ has $s_i + 1$ children and any descendant $j$ of $i$ satisfies $j < i$.

![Diagram of an $s$-decreasing tree]

**Figure:** An $(0, 0, 2, 1, 3)$-decreasing tree.
Let $s = (s_1, \ldots, s_n)$ be a composition (i.e. $s_i \in \mathbb{N}_{>0}$).

An $s$-decreasing tree is associated to a multipermutation of $1^{s_1} \ldots n^{s_n}$ that avoids the pattern 121. Such multipermutations are called *Stirling s-permutations*.

![Diagram of an s-decreasing tree and corresponding Stirling s-permutation](image)

**Figure:** An $(1, 1, 2, 2)$-decreasing tree and the corresponding Stirling s-permutation 313442.
The $s$-weak order

Figure: The $(1, 2, 2)$-weak order.
Conjecture 1 (Ceballos-Pons ’19)

The $s$-permutahedron can be realized as a polyhedral subdivision of a polytope which is combinatorially isomorphic to a permutahedron.

Conjecture 2 (Ceballos-Pons ’19)

If $s$ has no zeros, there exists a geometric realization of the $s$-permutahedron such that the $s$-associahedron can be obtained from it by removing certain facets.
Outline

1. The $s$-weak order and the $s$-permutahedron

2. Triangulation of a flow polytope

3. Mixed subdivision of a sum of hypercubes

4. Polytopal subdivision delimited by an arrangement of tropical hypersurfaces
Associated to a composition $s = (s_1, \ldots, s_n)$ we consider the graph $G_s$ on vertices $v_0, v_1, \ldots, v_{n+1}$ with:
- two edges from $v_i$ to $v_{i+1}$ for $i \in [n]$ and one edge from $v_0$ to $v_1$,
- $s_{n+1-i} - 1$ edges from $v_0$ to $v_i$ for $i \in [n]$,
- the **framing** given by ordering incoming and outgoing edges from top to bottom on the drawing.

A **route** is a path from $v_0$ to $v_{n+1}$.

The **flow polytope** $F_{G_s} = \left\{ (f_e)_{e \in E} \text{ flow of } G \right\} \subset \mathbb{R}^E$ is the convex hull of the indicator vectors of the routes of $G_s$. 

![Graph Diagram]

$s = 132$
We say that two routes $P, Q$ of $G$ are coherent with respect to the framing if they "do not cross".

For $C \in C^{\text{max}}$ (set of maximal cliques of coherent routes), $\Delta_C$ denotes the simplex with vertices the indicator vectors of the routes in $C$.

**Theorem (Danilov-Karzanov-Koshevoy, '12)**

The simplices $\{\Delta_C | C \in C^{\text{max}}(G, \preceq)\}$ form a (regular) triangulation of $\mathcal{F}_G$, called the DKK triangulation of $\mathcal{F}_G$ with respect to the framing $\preceq$.

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**Figure:** The maximal clique $\{l^0_w, \ldots, l^4_w\}$ corresponding to the $(1, 2, 1)$-Stirling permutation $w = 3221$. 

\[ l^0_w, l^1_w, l^2_w, l^3_w, l^4_w \]
Theorem (GMPTY, ’22)

The $s$-decreasing trees are in bijection with the simplices of the DKK triangulation of $(\mathcal{F}_G, \preceq)$.
Moreover, two simplices are adjacent if and only if there is a cover relation in the $s$-weak order for the corresponding $s$-decreasing trees.

Figure: Dual of the DKK triangulation for $s = (1, 2, 1)$.
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Minkowski sums

- Given polytopes $P_1, \ldots, P_k$ in $\mathbb{R}^n$, their **Minkowski sum** is the polytope $P_1 + \ldots + P_k := \{ \sum x_i \mid x_i \in P_i \}$.
- The **Minkowski cells** of the sum are $\sum B_i$ where $B_i$ is the convex hull of a subset of vertices of $P_i$.
- A **mixed subdivision** of a Minkowski sum is a collection of Minkowski cells such that their union covers the Minkowski sum and they intersect properly.
- A **fine mixed subdivision** is a minimal mixed subdivision via containment.

**Figure**: A (non fine) mixed subdivision of a sum of a square and a triangle.

Credit: De Loera-Rambau-Santos '19
Cayley Trick

\[ C(P_1, \ldots, P_k) := \text{conv}(\{e_1\} \times P_1, \ldots, \{e_k\} \times P_k) \subset \mathbb{R}^k \times \mathbb{R}^n \] is the Cayley embedding of \( P_1, \ldots, P_k \).

Proposition (The Cayley trick)

The (regular) polytopal subdivisions (resp. triangulations) of \( C(P_1, \ldots, P_k) \) are in bijection with the (coherent) mixed subdivisions (resp. fine mixed subdivisions) of \( P_1 + \ldots + P_k \).

Credit: De Loera-Rambau-Santos '19
Flow polytopes are Cayley embeddings

**Theorem (GMPTY, ’22)**

The $s$-decreasing trees are in bijection with the maximal cells of a fine mixed subdivision of the Minkowski sum of hypercubes in $\mathbb{R}^{n-1}$ given by

$$(s_n + 1)\square_{n-1} + \sum_{i=1}^{n-1} (s_i - 1)\square_{i-1}.$$ 

Proof : The flow polytope of $G_s$ is a Cayley embedding of hypercubes.
Mixed subdivision of hypercubes

Figure: (a) Summands of the Minkowski cell corresponding to $w = 3221$. (b) Mixed subdivision of $2\square_2 + \square_1$ realizing the $(1, 2, 1)$-permutahedron.
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From the mixed subdivision to a dual polyhedral complex
The regular subdivision $S$ of a point configuration $A \subset \mathbb{R}^n$ can be obtained as the lower faces of the points of $A$ lifted by an \textit{admissible height function} $\alpha$.

Danilov-Karzanov-Koshevoy give an explicit admissible height function for DKK triangulations.
Such a lifted configuration is associated to a *tropical polynomial*:

\[
F(x) = \bigoplus_{i \in [m]} \alpha^i \odot x^{a^i} = \min \{ \alpha^i + \langle a^i, x \rangle \mid i \in [m] \},
\]

that defines the *tropical hypersurface*:

\[
\mathcal{T}(F) := \{ x \in \mathbb{R}^n \mid \text{the minimum of } F(x) \text{ is attained at least twice}\}.
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$$F(x) = \bigoplus_{i \in [m]} \alpha^i \circ x^a^i = \min \left\{ \alpha^i + \langle a^i, x \rangle \mid i \in [m] \right\},$$

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\]

that defines the *tropical hypersurface*:

\[
T(F) := \{ x \in \mathbb{R}^n \mid \text{the minimum of } F(x) \text{ is attained at least twice} \}.
\]
Such a lifted configuration is associated to a tropical polynomial:

\[
F(x) = \bigoplus_{i \in [m]} \alpha^i \odot x^{a^i} = \min \{ \alpha^i + \langle a^i, x \rangle \mid i \in [m] \},
\]

that defines the tropical hypersurface:

\[
\mathcal{T}(F) := \{ x \in \mathbb{R}^n \mid \text{the minimum of } F(x) \text{ is attained at least twice} \}.
\]

**Theorem (folklore)**

There is a bijection between the \( k \)-dimensional cells of \( S \) and the \( (n - k) \)-dimensional cells of \( \mathcal{T}(F) \).

The bounded cells of \( \mathcal{T}(F) \) corresponds to the interior cells of \( S \).
When the point configuration is a Cayley embedding, there is a factorization of the tropical polynomial of the mixed subdivision corresponding to $S$ via the Cayley trick and we obtain an arrangement of tropical hypersurfaces.

**Theorem (GMPTY, '22)**

The $s$-permutahedron can be realized as the bounded cells of an arrangement of tropical hypersurfaces.

We have explicit coordinates for the vertices and all the faces!
\[ s = 1114 \]
\[ s = 1333 \]
Thank you for your attention!


M. Joswig, "Essentials of tropical combinatorics", vol. 219, American Mathematical Society, 2021
