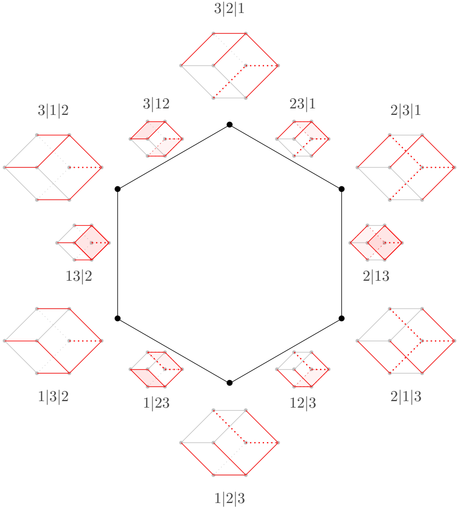
Pivot polytope of product of simplicies

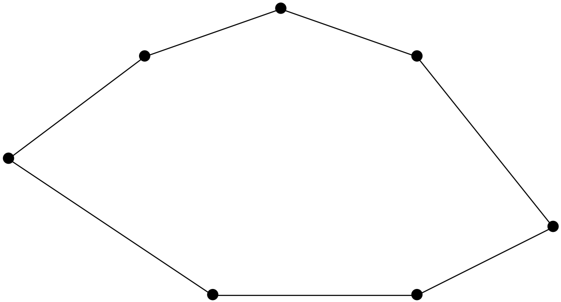
Vincent Pilaud, **Germain Poullot** & Raman Sanyal



Pivot rules and pivot rule polytopes

Shadow vertex rule

Linear optimization in dimension 2 (simplex method):



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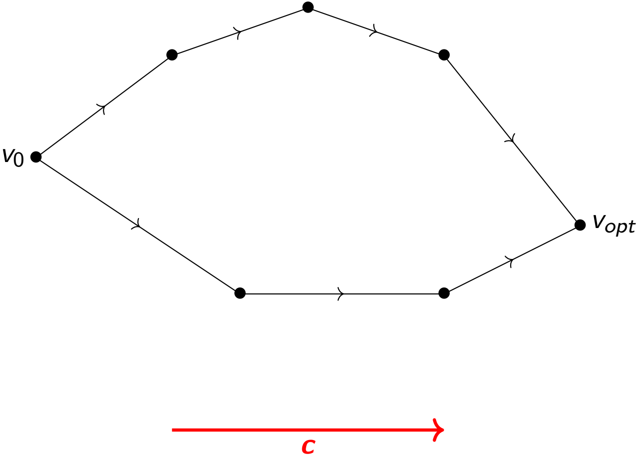

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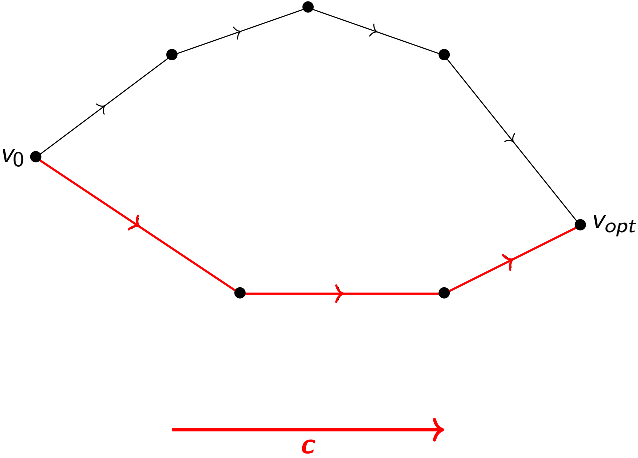

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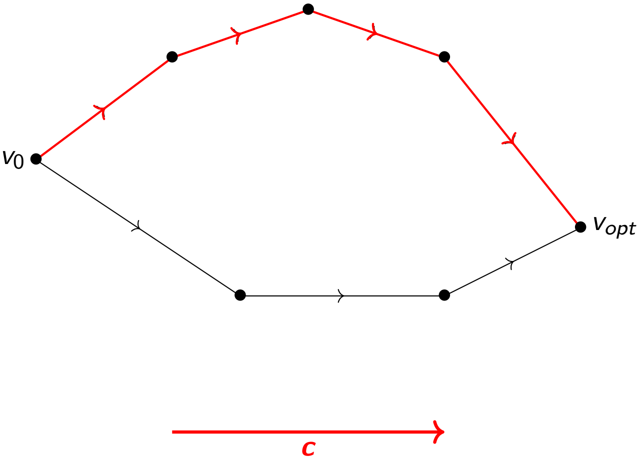
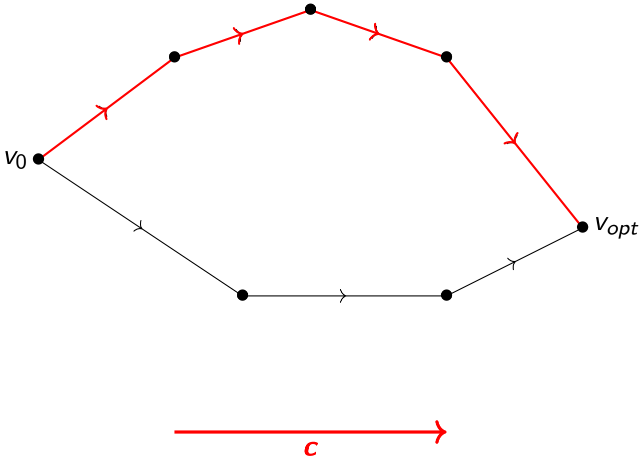


Shadow vertex rule

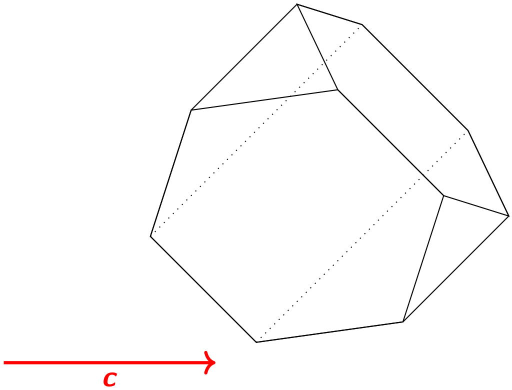
Linear optimization in dimension 2 (simplex method): **EASY** !



By convention, we always choose the upper path when optimizing.

Shadow vertex rule

Optimization in higher dimension: make it 2-dimensional !



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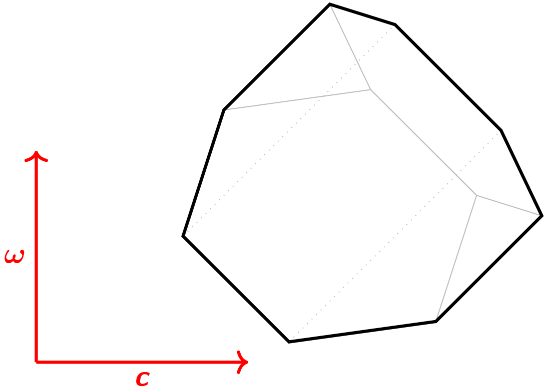

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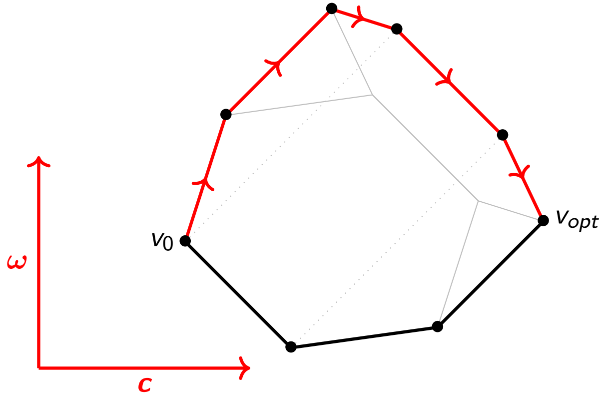

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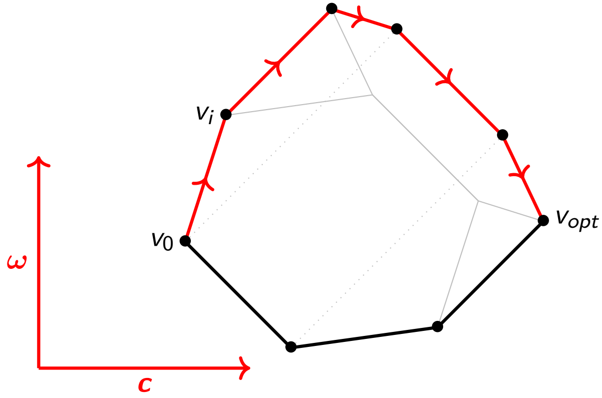

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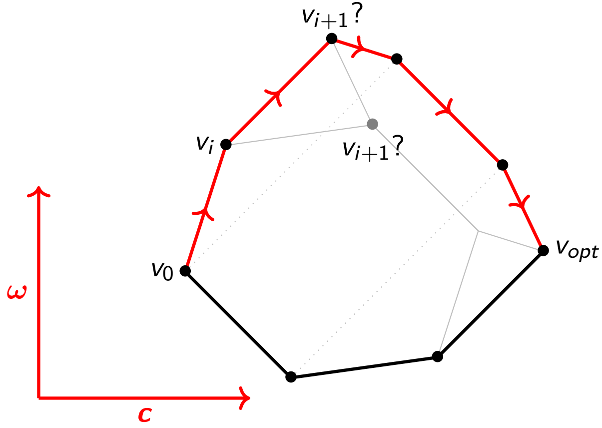

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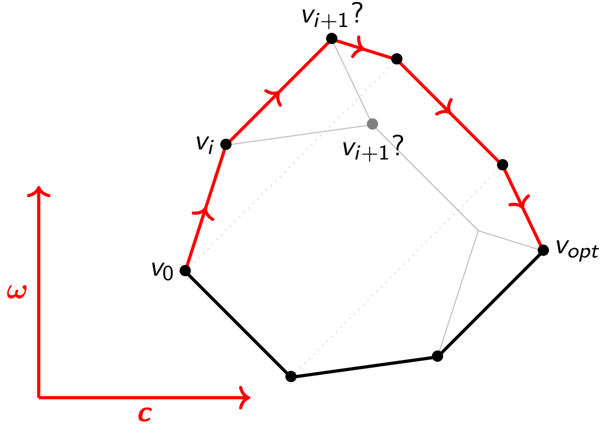

Shadow vertex rule (i.e. "take the neighbor with the best slope"):

$$A^\omega(v) = \text{argmax}\left\{\frac{\langle \boldsymbol{\omega}, u - v\rangle}{\langle \boldsymbol{c}, u - v\rangle}; u \text{ improving neighbor of } v\right\}$$

Shadow vertex rule

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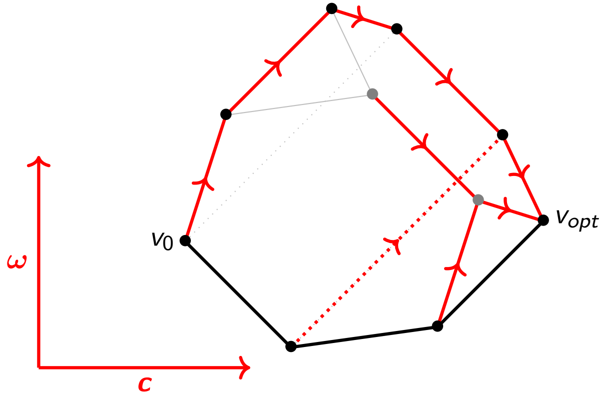

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Applying the rule at every vertex gives a *monotone arborescence*.

Monotone path polytope and pivot rule polytope

Let $\mathrm{P} \subset \mathbb{R}^d$ be a polytope.
Shadow vertex rule: $A^{\boldsymbol{\omega}}(v) = \operatorname{argmax}\left\{\frac{\langle \boldsymbol{\omega}, u-v\rangle}{\langle \boldsymbol{c}, u-v\rangle}; u \text{ impr. neig. of } v\right\}$.

Coherent monotone path: A monotone path that can be obtained via the shadow vertex rule.

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Monotone path polytope $\Sigma_{\boldsymbol{c}}(\mathrm{P})$ [BS92]: Fiber polytope of $\mathrm{P} \xrightarrow{\pi} \mathrm{Q}$ with Q a segment. (Can be seen as a Minkowski sum of sections of P.)
The vertices of $\Sigma_{\boldsymbol{c}}(\mathrm{P})$ are all coherent monotone paths.

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Pivot rule polytope $\Pi_{c}(\mathrm{P})$: Polytope which vertices are all coherent arborescences.

Monotone path polytope and pivot rule polytope

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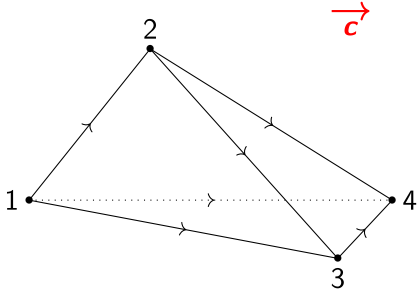
Coherent arborescence: An arborescence that can be obtained via the shadow vertex rule.
Pivot rule polytope $\Pi_{\boldsymbol{c}}(\mathrm{P})$: Polytope which vertices are all coherent arborescences.

$$\Pi_{\boldsymbol{c}}(\mathrm{P}) = \operatorname{conv}\left\{\sum_{v \neq v_{opt}} \frac{1}{\langle \boldsymbol{c}, A(v)-v\rangle}(A(v)-v); A \text{ coherent arbo. of P}\right\}$$

Case of the d-simplex



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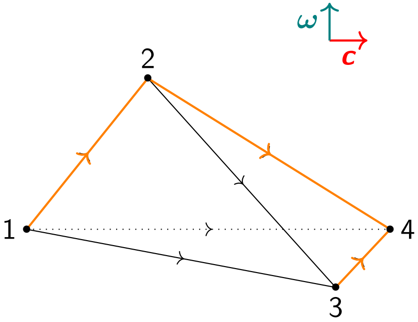

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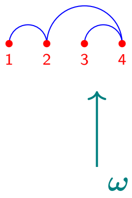

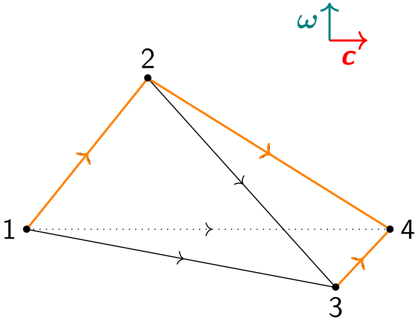

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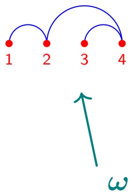

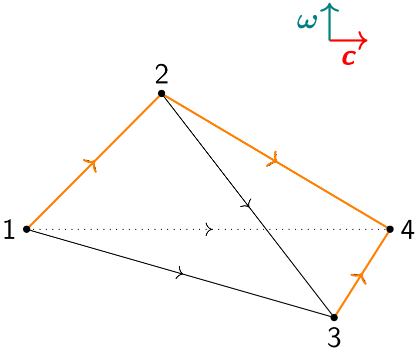

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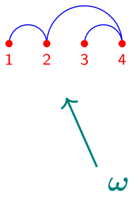

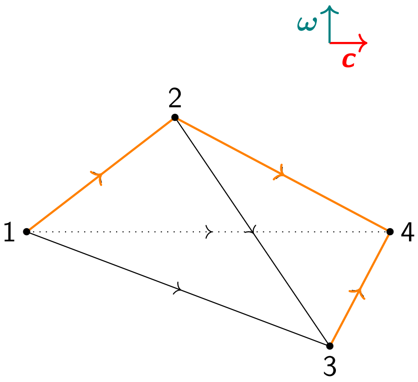

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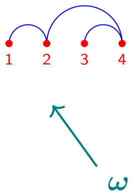

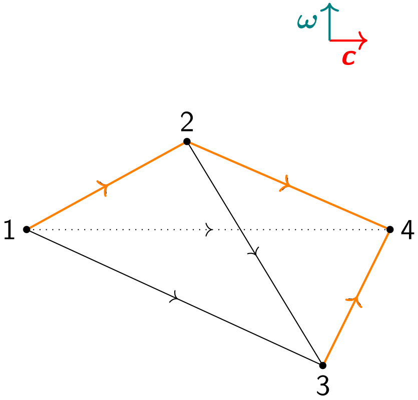

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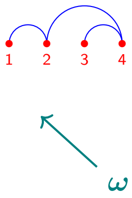

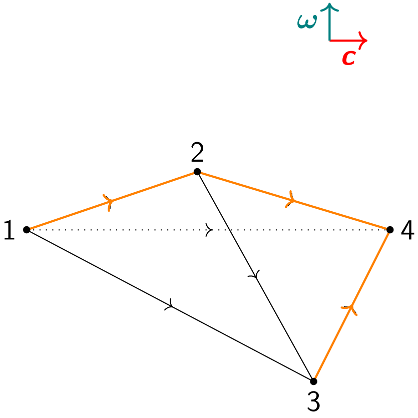

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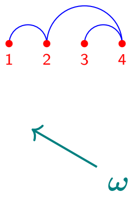

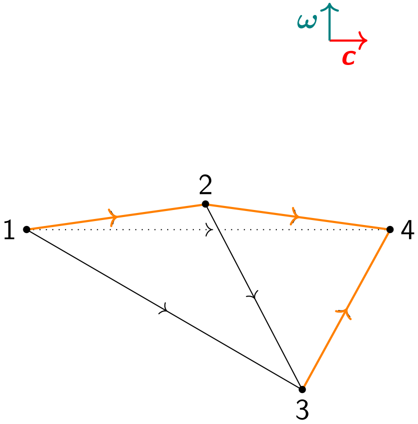

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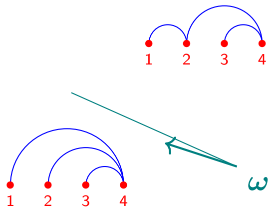

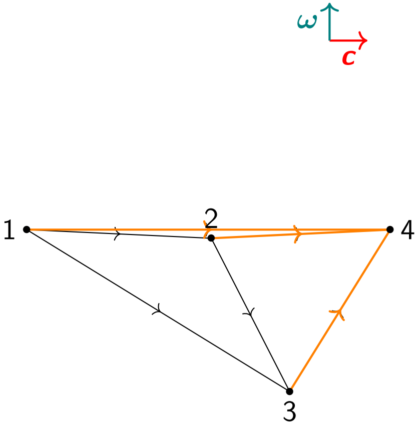

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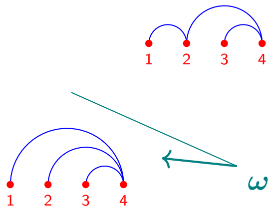

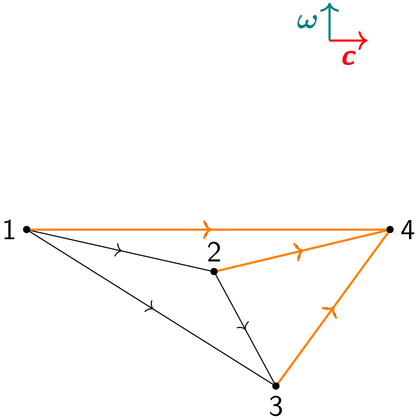

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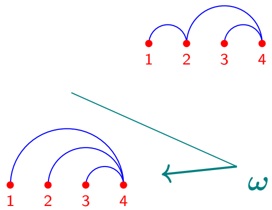

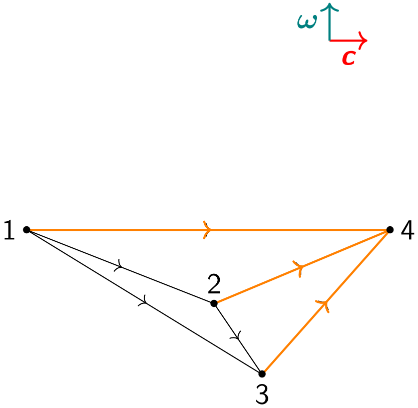

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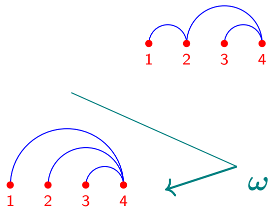

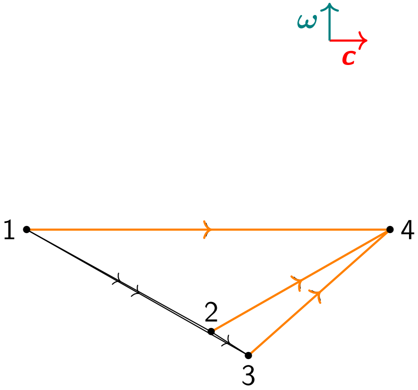

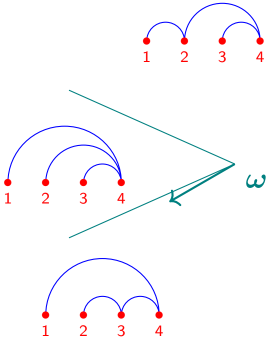
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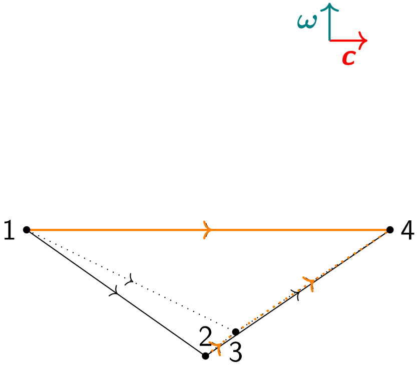
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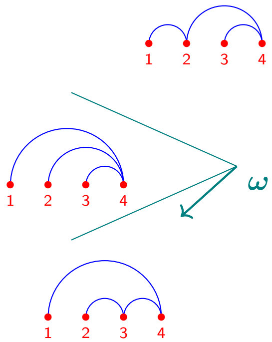
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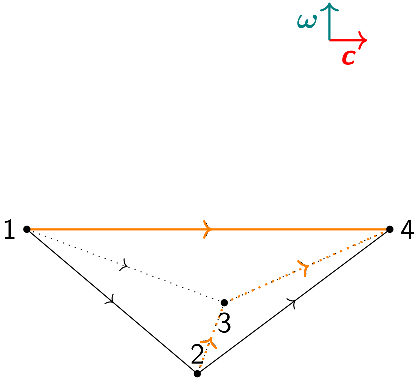
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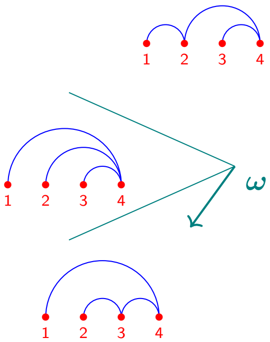

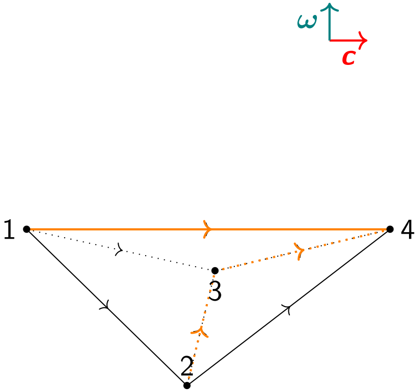

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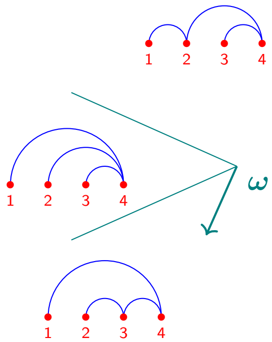

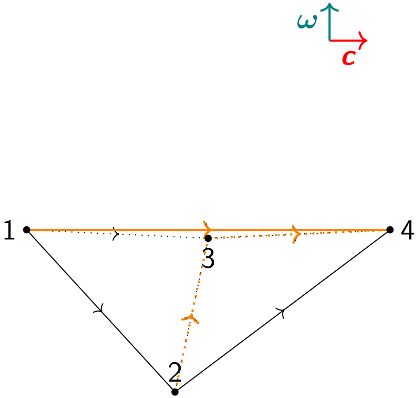

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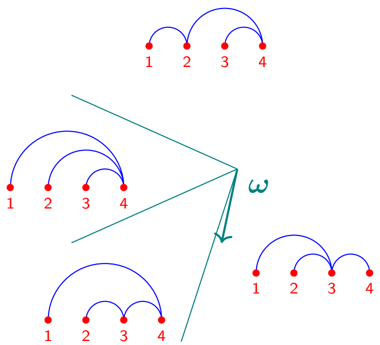

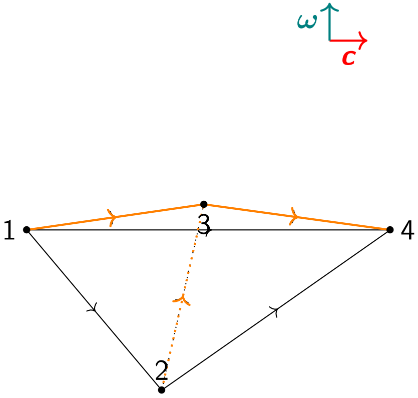

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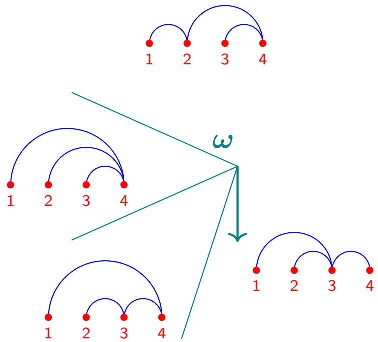

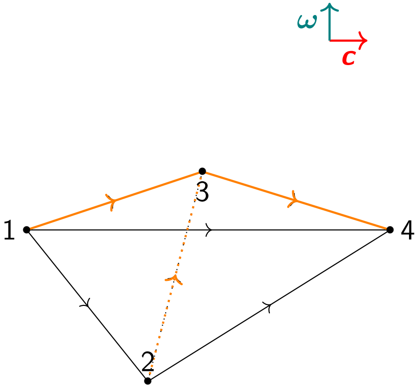

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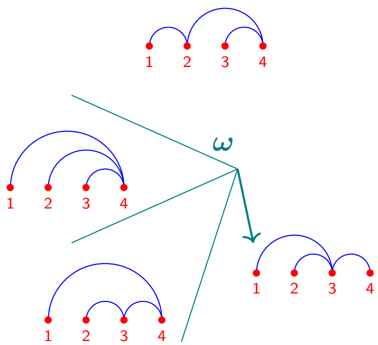

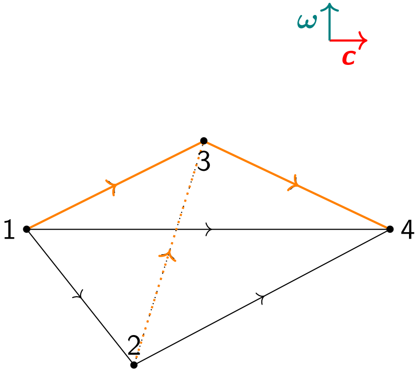

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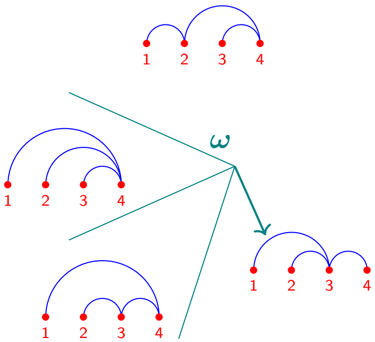

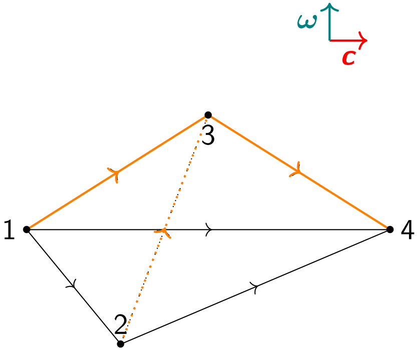

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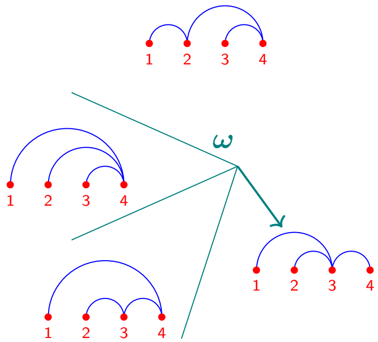

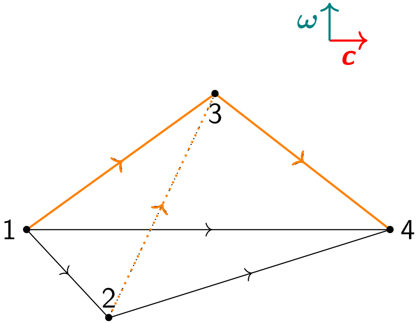

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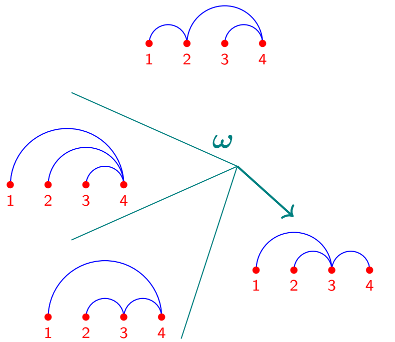

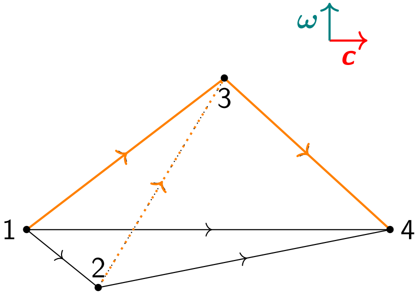

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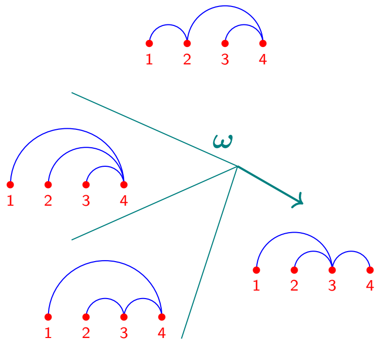

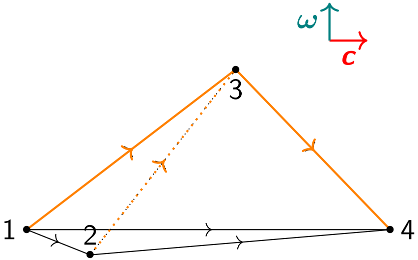

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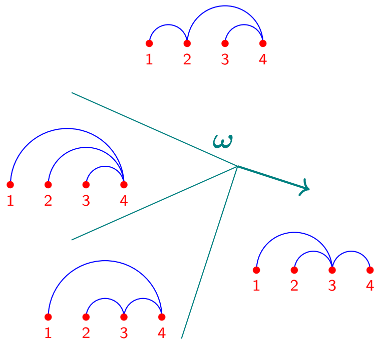

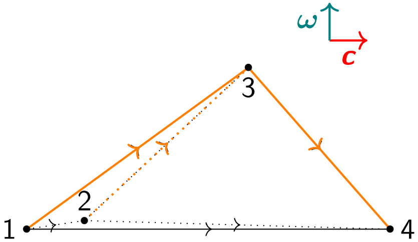

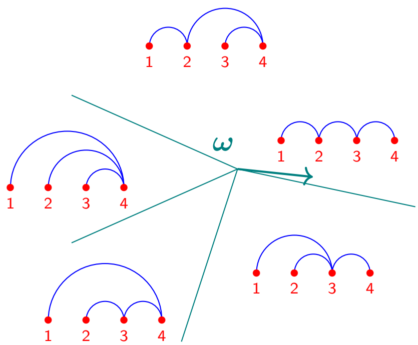
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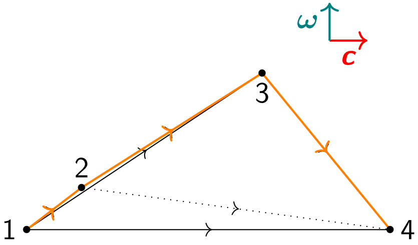
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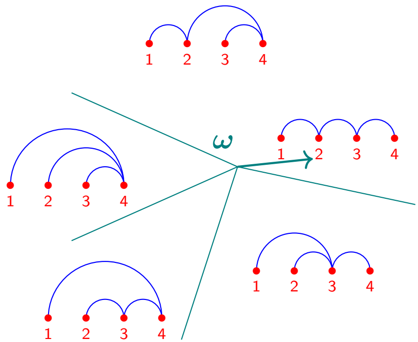
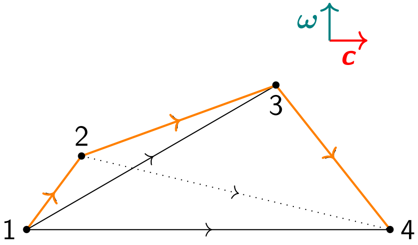
Case of the d-simplex





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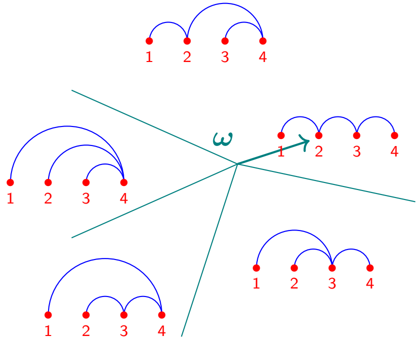

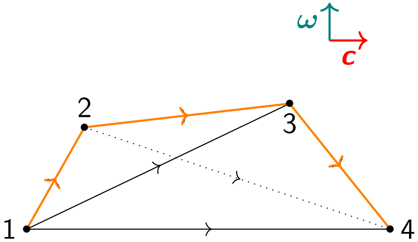

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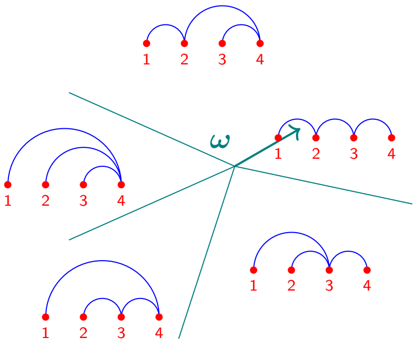
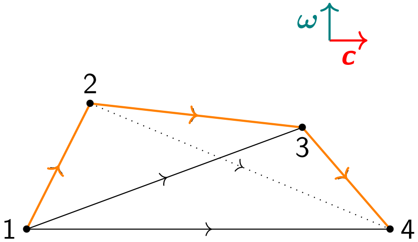
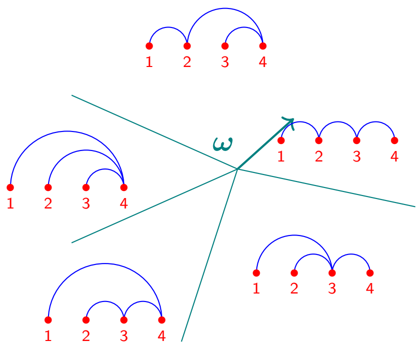
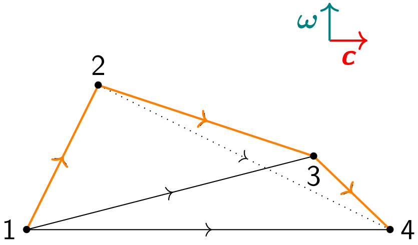





1 2 3 4
ω
1 2 3 4
1 2 3 4
1 2 3 4
1 2 3 4


ω
c
1
2
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4

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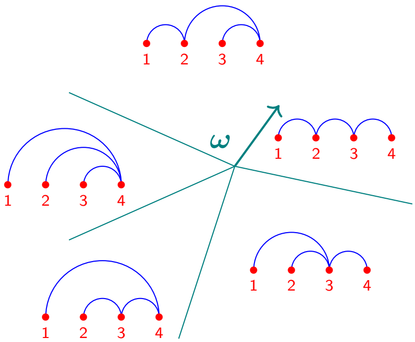

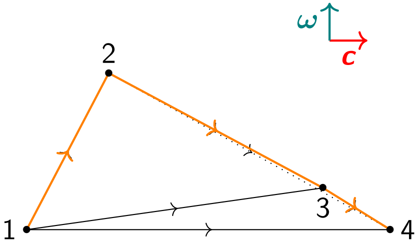

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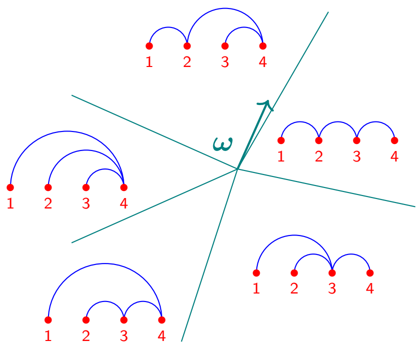

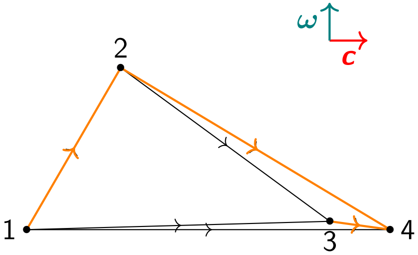

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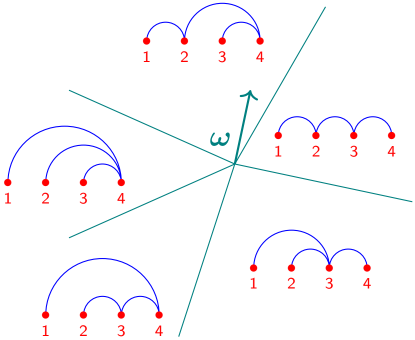

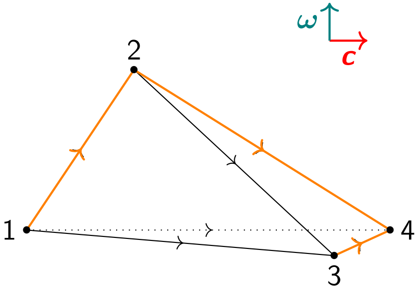

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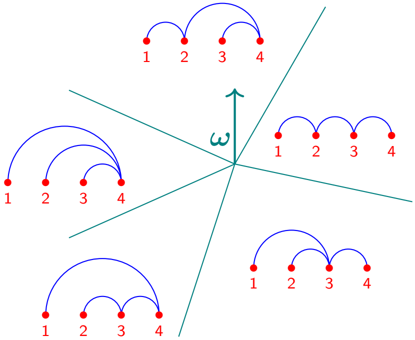

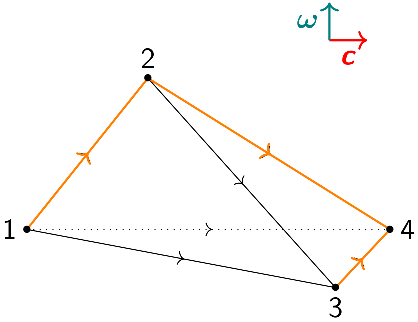

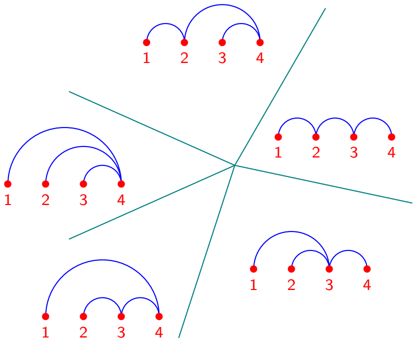

Pivot rule fan $\pi_{\boldsymbol{c}}(\mathrm{P})$:
$\boldsymbol{\omega} \sim \boldsymbol{\omega}'$ iff $A^{\boldsymbol{\omega}} = A^{\boldsymbol{\omega}'}$.
This gives a polytopal fan [BDLLS22] (see above).

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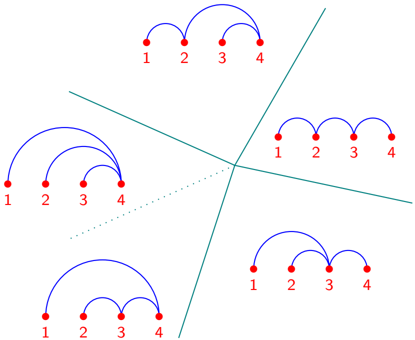

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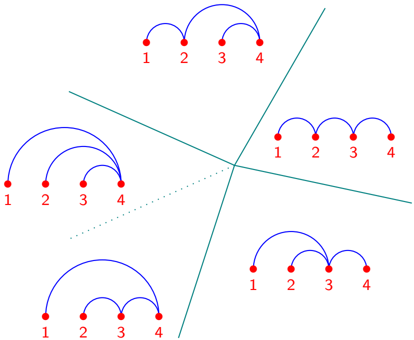

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$$\Sigma_{\boldsymbol{c}}(\Delta_d) = \mathrm{Cube}_{d-1}$$
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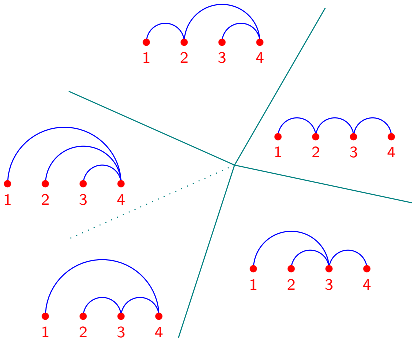

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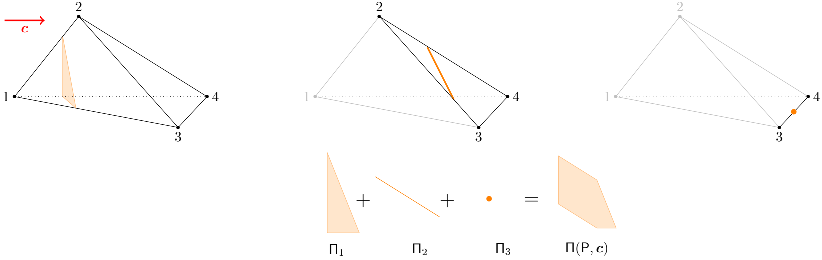
$\Sigma_{\boldsymbol{c}}(\Delta_d)$ [BS92]:
A monotone path $= (v_0,$ part of the vertices, $v_{opt})$.
Choosing a monotone path = Choosing a part of the $(d-1)$-remaining vertices.
Exercise: Prove all such paths are coherent.

Coherent arborescence: An arborescence that can be obtained via the shadow vertex rule.

Pivot rule polytope $\Pi_{\boldsymbol{c}}(\mathrm{P})$: Polytope which vertices are all coherent arborescences. Can also be seen as a Minkowski sum of sections:

$$\sum_{v \in V(\mathrm{P})} (\text{section between } v \text{ and its improving neighbors})$$



Poset of slopes

Slope comparisons

Fix P, $\boldsymbol{c}$. n vertices $V(\mathrm{P})$, m edges $E(\mathrm{P})$, dimension d.
Shadow vertex rule: $A^{\boldsymbol{\omega}}(v) = \operatorname{argmax}\left\{\frac{\langle\boldsymbol{\omega}, u-v\rangle}{\langle\boldsymbol{c}, u-v\rangle} ; u \text{ impr. neig. of } v\right\}$.

For $\boldsymbol{\omega}$, what is important? (to compute $A^{\boldsymbol{\omega}}$)

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For ω, what is important? (to compute A^{ω})

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ground set : $E(\mathrm{P})$
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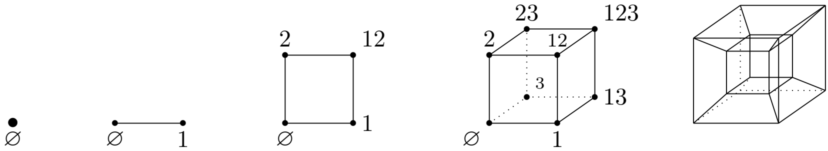
$\Rightarrow$ Where is $\theta(\boldsymbol{\omega})$ in the braid fan (i.e. compare its coordinates)?

Case of the d-cube



Cube: $\mathrm{P} = \square_d = [0,1]^d$

$d2^{d-1}$ edges

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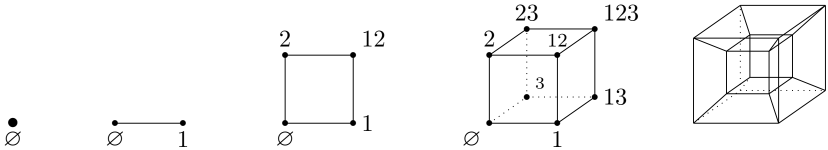

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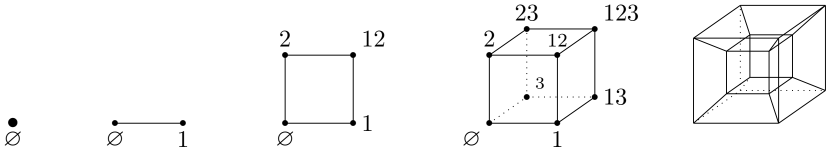

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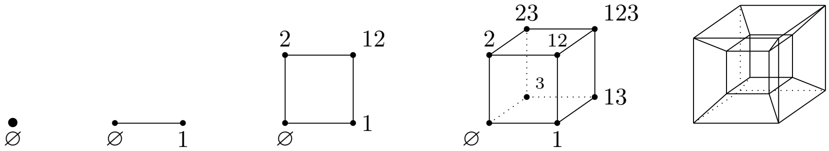

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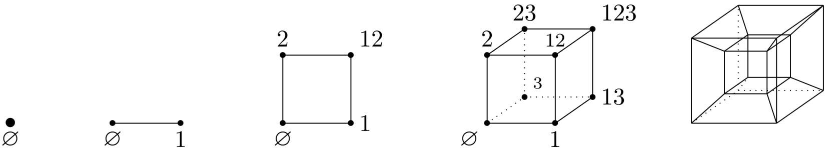

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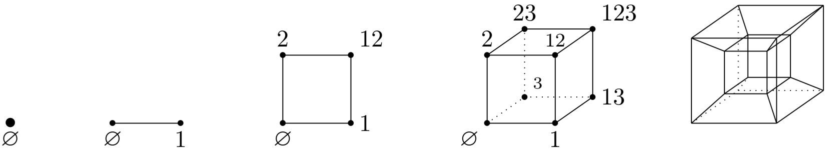

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and $\preceq_\omega$ is a permutation of $\{1, \dots, d\}$ that fully describes A^ω

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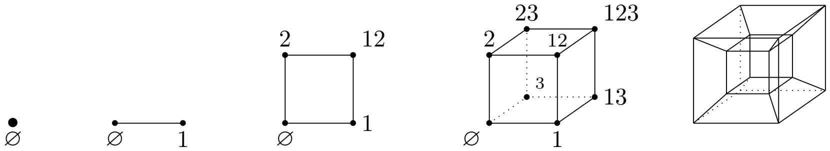

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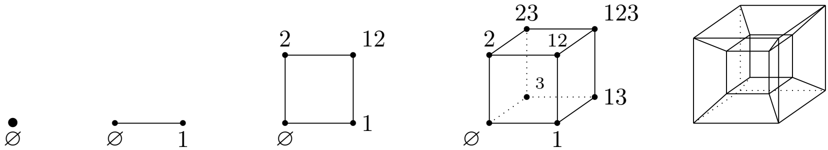

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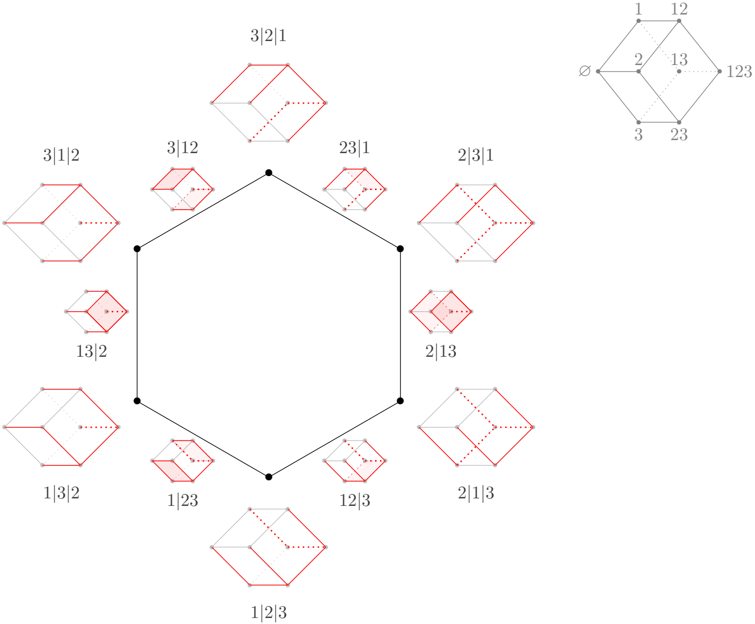
$\Rightarrow \Pi_{\boldsymbol{c}}(\square_d)$ is a permutahedron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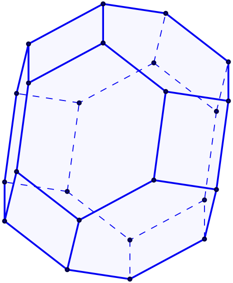
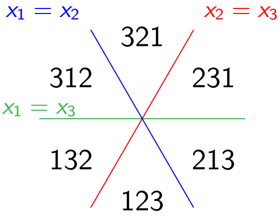
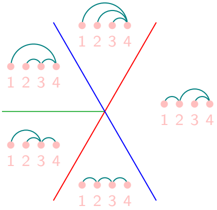
3|12
23|1
2|3|1
13|2
2|13
1|3|2
1|23
12|3
2|1|3
1|2|3

Braid fan: Fan of the hyperplane arrangement $H_{i,j} = \{\boldsymbol{x} \; ; \; x_i = x_j\}$







Coarsening: Choose maximal cones and merge them

Generalized permutahedra: P whose normal fan coarsens $\mathcal{B}_n$ (permutahedron, associahedron, cube, hypersimplex...), each face associates to a poset on $[n]$

$\mathcal{P}(\mathrm{P})$: all the posets associated to faces of P

Generalized permutahedra

Aim: Link pivot polytopes with generalized permutahedra.

Hint:

$$\Pi_{\boldsymbol{c}}(\square_d) = \mathrm{Perm}_d$$

$$\Pi_{\boldsymbol{c}}(\Delta_d) = \mathrm{Asso}_d$$

Comparison of slopes is comparison of coordinates $\Rightarrow$ braid fan

Mimicking the case of the d-cube

Idea 1:
Fix a polytope P, and direction $\boldsymbol{c}$, n vertices, m edges.

$\theta : \mathbb{R}^d \to \mathbb{R}^m$ sends the pivot fan inside $\text{Im}(\theta) \cap \mathcal{B}_m$

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We need to go lower dimensional!

Idea 2:
Fix a polytope P, direction $\boldsymbol{c}$,
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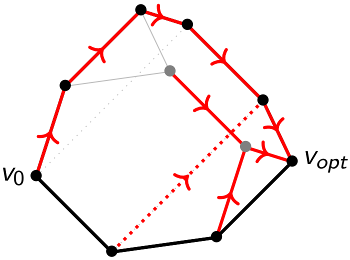
Fix A arborescence:
$\vartheta_A(\boldsymbol{\omega}) = (\tau_{\boldsymbol{\omega}}(u, A(u))\ ;\ u \text{ vertex})$



Adapted slope map

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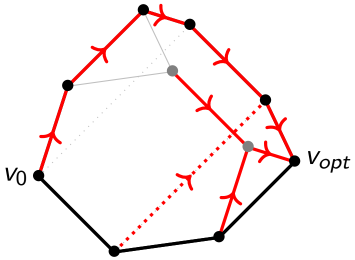

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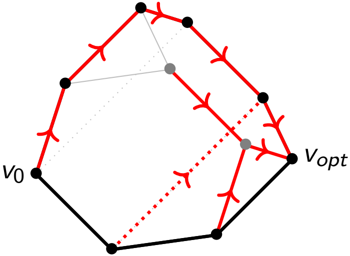

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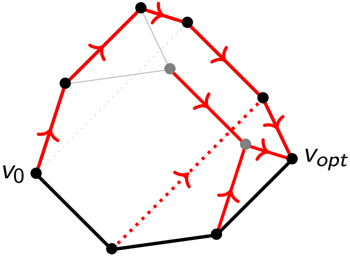

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What if $d = n - 1$?

Pivot rule polytope of products of simplices

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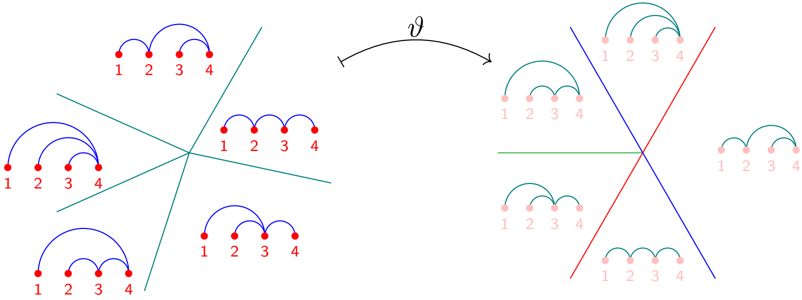

Theorem (Pivot polytope simplex)

For all simplex, all (generic) direction: $\Pi_{\boldsymbol{c}}(\Delta_d) \simeq \mathrm{Asso}_d$.

Already in [BDLLSon], but new proof.

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Pivot polytope of simplices and cubes

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Theorem (Pivot polytope standard cube)

For standard cubes, all (generic) direction: $\Pi_{\mathbf{c}}(\square_d) \simeq \mathrm{Perm}_d$.

Already in [BDLLS22], but new proof.

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Remark: $\square_d = [0,1]^d = \Delta_1 \times \cdots \times \Delta_1$.

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Now: identify the coarsening.

Shuffles

Shuffle: $(E, \leq)$ and $(F, \preceq)$ posets, then $\trianglelefteq$ is a shuffle when:

groud set : $E \sqcup F$

relations : all relations of $\leq$; all relations of $\preceq$;
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Theorem (Shuffle product [CP22])

P, Q*: generalized permutahedra. There exists polytope* $\mathsf{P} \star \mathsf{Q}$ *s.t.*
$\mathcal{P}(\mathsf{P} \star \mathsf{Q}) = \{$*all shuffles between* $\leq \in \mathcal{P}(\mathsf{P})$ *and* $\preceq \in \mathcal{P}(\mathsf{Q})\}$

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Theorem (Pivot polytope of products of simplices)

For $\Delta_{d_1} \times \cdots \times \Delta_{d_r}$, all (generic) direction, via $\overline{\vartheta}$:

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Example

(a) $\Pi_{\boldsymbol{c}}(\square_d) \simeq \mathrm{Perm}_d$
(b) $\Pi_{\boldsymbol{c}}(\square_m \times \Delta_n) \simeq (m, n)$-multiplihedron
(c) $\Pi_{\boldsymbol{c}}(\Delta_m \times \Delta_n) \simeq (m, n)$-constrainahedron

Conjecture and open problems

1) Is $\Pi_{\boldsymbol{c}}(\mathrm{P})$ projection of a generalized permutahedron?
$\longrightarrow$ pivot fan sent inside $\mathrm{Im}(\overline{\theta}) \cap \mathcal{B}_{m'}$

2) For which P, $\Pi_{\boldsymbol{c}}(\mathrm{P})$ is a generalized permutahedron?
$\longrightarrow$ a priori, only products of simplices, but no proof

3) When $\Pi_{\boldsymbol{c}}(\mathrm{P})$ and $\Pi_{\boldsymbol{c}}(\mathrm{Q})$ are **not** generalized permutahedra, then what happen to $\Pi_{\boldsymbol{c}}(\mathrm{P} \times \mathrm{Q})$?
$\longrightarrow$ not equivalent to $\Pi_{\boldsymbol{c}}(\mathrm{P}) \star \Pi_{\boldsymbol{c}}(\mathrm{Q})$, but "embeds" in it

Thank you!

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