Pivot polytope of product of simplicies

Vincent Pilaud, Germain Poullot & Raman Sanyal
1. Pivot rules and pivot rule polytopes

2. Poset of slopes

3. Pivot rule polytope of products of simplices
Pivot rules and pivot rule polytopes
Shadow vertex rule

Linear optimization in dimension 2 (simplex method):

By convention, we always choose the upper path when optimizing.
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\[ v_0 \rightarrow \ldots \rightarrow v_{\text{opt}} \]
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Linear optimization in dimension 2 (simplex method): EASY!

By convention, we always choose the upper path when optimizing.
Shadow vertex rule

Optimization in higher dimension: make it 2-dimensional!

Shadow vertex rule (i.e. “take the neighbor with the best slope”):

\[ \omega(v) = \arg\max \left\{ \langle \omega, u - v \rangle \mid \langle c, u - v \rangle \right\}; \] improving neighbor of \( v \)

Applying the rule at every vertex gives a monotone arborescence.
Shadow vertex rule

Optimization in higher dimension: make it 2-dimensional!

\[
\omega \cdot v_i + 1 \overset{\text{?}}{\overbrace{\omega \cdot v_i}}
\]

Shadow vertex rule (i.e. “take the neighbor with the best slope”):

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A_\omega(v) = \arg\max \left\{ \langle \omega, u - v \rangle, \langle c, u - v \rangle \right\}; \quad u \text{ improving neighbor of } v
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\]

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\[
A^ω(ν) = \arg\max \left\{ \frac{⟨ω, u − ν⟩}{⟨c, u − ν⟩} ; u \text{ improving neighbor of } ν \right\}
\]
Shadow vertex rule

Optimization in higher dimension: make it 2-dimensional!

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A^\omega(v) = \arg\max \frac{\langle \omega, u - v \rangle}{\langle c, u - v \rangle}; u \text{ improving neighbor of } v
\]

Applying the rule at every vertex gives a \textit{monotone arborescence}. 
Monotone path polytope and pivot rule polytope

Let $P \subset \mathbb{R}^d$ be a polytope.

Shadow vertex rule: $A^{\omega}(v) = \arg\max \left\{ \frac{\langle \omega, u-v \rangle}{\langle c, u-v \rangle}; u \text{ impr. neig. of } v \right\}$.

**Coherent monotone path**: A monotone path that can be obtained via the shadow vertex rule.
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**Monotone path polytope** $\Sigma_c(P)$ [BS92]: Fiber polytope of $P \xrightarrow{\pi} Q$ with $Q$ a segment. (Can be seen as a Minkowski sum of sections of $P$.) The vertices of $\Sigma_c(P)$ are all coherent monotone paths.
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**Pivot rule polytope $\Pi_c(P)$**: Polytope which vertices are all coherent arborescences.

$$\Pi_c(P) = \text{conv} \left\{ \sum_{v \neq v_{\text{opt}}} \frac{1}{\langle c, A(v) - v \rangle} (A(v) - v); A \text{ coherent arbo. of } P \right\}$$
Case of the $d$-simplex
Case of the $d$-simplex

\[ \omega \]

\[ \omega \]

\[ c \]

\[ 1 \]

\[ 2 \]

\[ 3 \]

\[ 4 \]
Case of the $d$-simplex
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Pivot polytope of product of simplicies
Case of the $d$-simplex

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\]

\[
\omega
\]

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\]

\[
\omega
\]
Case of the $d$-simplex
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Pivot polytope of product of simplices 7 / 24
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Case of the $d$-simplex

\[ \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array} \]
Case of the $d$-simplex
Case of the $d$-simplex
Case of the $d$-simplex
Case of the $d$-simplex

\begin{align*}
\omega & = (1, 2, 3, 4) \\
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\end{align*}
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\[ \omega \]

\[ \begin{array}{cccc}
1 & 2 & 3 & 4 \\
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\end{array} \]
Case of the $d$-simplex
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\[ \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array} \]

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\[ \begin{align*}
\omega &\cdot 1 \\
\omega &\cdot 2 \\
\omega &\cdot 3 \\
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1 &\cdot 1 \\
1 &\cdot 2 \\
1 &\cdot 3 \\
1 &\cdot 4 \\
2 &\cdot 1 \\
2 &\cdot 2 \\
2 &\cdot 3 \\
2 &\cdot 4 \\
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Pivot polytope of product of simplices
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**Pivot rule fan** $\pi_c(P)$:

$\omega \sim \omega'$ iff $A^\omega = A^{\omega'}$.

This gives a polytopal fan [BDLLS22] (see above).
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The pivot rule fan refines the monotone path fan.
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For any $d$-simplex $\Delta_d$, any $c$:

$$\Sigma_c(\Delta_d) = \text{Cube}_{d-1}$$
$$\Pi_c(\Delta_d) = \text{Asso}_d$$
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$\Sigma_c(\Delta_d)$ [BS92]:

A monotone path $= (v_0, \text{part of the vertices}, v_{opt})$.

Choosing a monotone path $= $ Choosing a part of the $(d-1)$-remaining vertices.

Exercise: Prove all such paths are coherent.
**Coherent arborescence**: An arborescence that can be obtained via the shadow vertex rule.

**Pivot rule polytope** $\Pi_c(P)$: Polytope which vertices are all coherent arborescences. Can also be seen as a Minkowski sum of sections:

$$\sum_{v \in V(P)} (\text{section between } v \text{ and its improving neighbors})$$
Poset of slopes
Fix \( P \), \( c \). \( n \) vertices \( V(P) \), \( m \) edges \( E(P) \), dimension \( d \).
Shadow vertex rule: \( A^\omega(v) = \arg\max \left\{ \frac{\langle \omega, u-v \rangle}{\langle c, u-v \rangle}; \ u \ \text{impr. neig. of} \ v \right\} \).

For \( \omega \), what is important? (to compute \( A^\omega \))
Fix $P$, $c$. $n$ vertices $V(P)$, $m$ edges $E(P)$, dimension $d$.

Shadow vertex rule: $A^\omega(v) = \text{argmax} \left\{ \frac{\langle \omega, u-v \rangle}{\langle c, u-v \rangle}; u \text{ impr. neig. of } v \right\}$.

For $\omega$, what is important? (to compute $A^\omega$)

The \textit{slopes}: $\tau^\omega(u, v) = \frac{\langle \omega, u-v \rangle}{\langle c, u-v \rangle}$
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$\Rightarrow$ \textit{Slope vector}: $\theta(\omega) = (\tau_\omega(u, v); uv \text{ improving edge of } P) \in \mathbb{R}^m$
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$\theta$ is a linear map $\mathbb{R}^d \to \mathbb{R}^m$, injective
Slope comparisons

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What is really important?? The comparisons of slopes!
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What is really important?? The comparisons of slopes!

Slope pre-order of $\omega$:

ground set : $E(P)$
relations : $uv \preceq_\omega u'v'$ $\iff$ $\tau_\omega(u, v) \leq \tau_\omega(u', v')$
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Slope pre-order of $\omega$:

- ground set: $E(P)$
- relations: $uv \preceq^\omega u'v' \iff \tau_\omega(u, v) \leq \tau_\omega(u', v')$

$\Rightarrow$ Where is $\theta(\omega)$ in the braid fan (i.e. compare its coordinates)?
Case of the $d$-cube

Cube: $P = \Box_d = [0, 1]^d$

$d2^{d-1}$ edges
Case of the $d$-cube

Cube: $P = \square_d = [0, 1]^d$

$d2^{d-1}$ edges, **but** $d$ classes of parallelism of edges!
Case of the $d$-cube

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$\overline{\theta}(\omega) = \theta(\omega)$ restricted to $d$ non-parallel edges
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$\bar{\theta} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ linear
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$\overline{\theta} : \mathbb{R}^d \to \mathbb{R}^d$ linear + injective $\Rightarrow$ automorphism, and $\preceq_\omega$ is a permutation of $\{1, \ldots, d\}$ that fully describes $A^\omega$
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Moreover, bijection: $A^\omega \leftrightarrow$ permutations $\{1, \ldots, d\}$
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$\Rightarrow \Pi_c(\Box_d)$ is a permutahedron
Case of the $d$-cube
**Generalized permutahedra**

*Braid fan*: Fan of the hyperplane arrangement $H_{i,j} = \{ x ; x_i = x_j \}$

![Diagram of braid fan]

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**Coarsening**: Choose maximal cones and merge them

**Generalized permutahedra**: $P$ whose normal fan coarsens $B_n$ (permutahedron, associahedron, cube, hypersimplex...), each face associates to a poset on $[n]$

$\mathcal{P}(P)$: all the posets associated to faces of $P$
Aim: Link pivot polytopes with generalized permutahedra.

Hint:

\[ \Pi_c(\square_d) = \text{Perm}_d \]
\[ \Pi_c(\Delta_d) = \text{Asso}_d \]

Comparison of slopes is comparison of coordinates \( \Rightarrow \) braid fan
Idea 1:
Fix a polytope $P$, and direction $c$, $n$ vertices, $m$ edges.

$\theta : \mathbb{R}^d \to \mathbb{R}^m$ sends the pivot fan inside $\text{Im}(\theta) \cap \mathcal{B}_m$
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Problem: This is not a braid fan as $d << m$...
Mimicking the case of the \( d \)-cube

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\textit{Problem:} This is not a braid fan as \( d << m \)...

If \( m' \) classes of parallelism:

\( \overline{\theta} : \mathbb{R}^d \to \mathbb{R}^{m'} \) sends the pivot fan inside \( \text{Im}(\theta) \cap \mathcal{B}_{m'} \)
Mimicking the case of the $d$-cube

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Problem: This is not a braid fan as $d << m' < m$...

We need to go lower dimensional!
Idea 2:
Fix a polytope $P$, direction $c$, $n$ vertices, $m$ edges.
Fix $A$ arborescence:
$\vartheta_A(\omega) = (\tau_\omega(u, A(u)) ; u \text{ vertex})$
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$\vartheta_A$: linear, injective, $\mathbb{R}^d \rightarrow \mathbb{R}^{n-1}$

**but** if $\omega$ does not capture $A$, then $\vartheta_A(\omega)$ have no meaning...
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**Adapted slope map**: $\vartheta(\omega) = \vartheta_A(\omega)$
i.e. take $\omega$ and look at the slope of the edges it selects.
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Adapted slope map:
$$
\vartheta(\omega) = \vartheta_A \omega(\omega)
$$
i.e. take $\omega$ and look at the slope of the edges it selects.

$\vartheta$: piece-wise linear, injective, $\mathbb{R}^d \rightarrow \mathbb{R}^{n-1}$
i.e. $\vartheta$ sends the pivot fan inside $\text{Im}(\vartheta) \cap \mathcal{B}_{n-1}$

What if $d = n - 1$?
Pivot rule polytope of products of simplices
Case of the $d$-simplex

\[ d = n - 1 \iff \text{P is a simplex} \]
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\[ d = n - 1 \iff P \text{ is a simplex} \]

For $\Delta_d$: $\vartheta : \mathbb{R}^d \to \mathbb{R}^d$ piece-wise linear, $\ker \vartheta = \{0\}$
Case of the $d$-simplex

\[ d = n - 1 \iff P \text{ is a simplex} \]

For $\Delta_d$: $\vartheta : \mathbb{R}^d \rightarrow \mathbb{R}^d$ piece-wise linear, $\ker \vartheta = \{0\} \Rightarrow \text{bijection}$
Case of the \( d \)-simplex

\[
d = n - 1 \iff \text{P is a simplex}
\]

For \( \Delta_d \): \( \vartheta : \mathbb{R}^d \to \mathbb{R}^d \) piece-wise linear, \( \ker \vartheta = \{0\} \Rightarrow \) bijection \( \vartheta \) sends the pivot fan of \( \Delta_d \) inside \( B_d \).
Theorem (Pivot polytope simplex)

For all simplex, all (generic) direction: $\Pi_c(\Delta_d) \simeq \text{Asso}_d$.

Already in [BDLLSon], but new proof.

Proof
1) $\vartheta$ is piece-wise linear & bijective: pivot fan corsens $B_d$. 
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Theorem (Pivot polytope standard cube)

For standard cubes, all (generic) direction: \( \Pi_c(\mathbf{d}) \cong \text{Perm}_d \).

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Theorem (Pivot polytope standard cube)

For standard cubes, all (generic) direction: $\Pi_c(\Box_d) \simeq \text{Perm}_d$.

Already in [BDLLS22], but new proof.
Remark: $\square_d = [0, 1]^d = \Delta_1 \times \cdots \times \Delta_1$. 
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What if \( P = \Delta_{d_1} \times \cdots \times \Delta_{d_r} \)?
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**Problem:** \( \vartheta : \mathbb{R}^d \to \mathbb{R}^{n-1} \), but \( n = \prod_i (d_i + 1) \neq d + 1 \)
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Quotient by parallelisms: \( \overline{\vartheta} = \vartheta \) restricted to parallelism classes

\( \overline{\vartheta} : \mathbb{R}^d \to \mathbb{R}^d \), piece-wise linear, \( \ker \overline{\vartheta} = \{0\} \), \( \Rightarrow \) bijective
Remark: $\square_d = [0, 1]^d = \Delta_1 \times \cdots \times \Delta_1$.

What if $P = \Delta_{d_1} \times \cdots \times \Delta_{d_r}$?

**Problem:** $\vartheta : \mathbb{R}^d \rightarrow \mathbb{R}^{n-1}$, **but** $n = \prod_i (d_i + 1) \neq d + 1$

Quotient by parallelisms: $\overline{\vartheta} = \vartheta$ restricted to parallelism classes

$\overline{\vartheta} : \mathbb{R}^d \rightarrow \mathbb{R}^d$, piece-wise linear, ker $\overline{\vartheta} = \{0\}, \Rightarrow$ bijective

**Lemma (First conclusion)**

$\overline{\vartheta}$ sends pivot fan of $\Delta_{d_1} \times \cdots \times \Delta_{d_r}$ inside $\mathcal{B}_d$, i.e. 

$\Pi_c(\Delta_{d_1} \times \cdots \times \Delta_{d_r})$ is a generalized permutahedra.
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**Lemma (First conclusion)**

$\overline{\vartheta}$ sends pivot fan of $\Delta_{d_1} \times \cdots \times \Delta_{d_r}$ inside $B_d$, i.e.

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Now: identify the coarsening.
**Shuffle**: \((E, \preceq)\) and \((F, \preceq)\) posets, then \(\preceq\) is a shuffle when:

- **ground set**: \(E \sqcup F\)
- **relations**: all relations of \(\preceq\); all relations of \(\preceq\);
- for each \(e \in E, f \in F\), choose if \(e \preceq f\) or \(e \succ f\)
- (+ transitive closure)
Shuffles: \((E, \leq)\) and \((F, \preceq)\) posets, then \(\triangleleft\) is a shuffle when:

- **ground set:** \(E \sqcup F\)
- **relations:** all relations of \(\leq\); all relations of \(\preceq\);
  for each \(e \in E, f \in F\), choose if \(e \triangleleft f\) or \(e \triangleright f\)
  (+ transitive closure)

**Theorem (Shuffle product [CP22])**

\(P, Q\): generalized permutahedra. There exists polytope \(P \star Q\) s.t.
\(\mathcal{P}(P \star Q) = \{\text{all shuffles between } \leq \in \mathcal{P}(P) \text{ and } \preceq \in \mathcal{P}(Q)\}\)
Theorem (Pivot polytope of products of simplices)

For $\Delta_{d_1} \times \cdots \times \Delta_{d_r}$, all (generic) direction, via $\vartheta$:

$$\Pi_c(\Delta_{d_1} \times \cdots \times \Delta_{d_r}) \simeq \text{Asso}_{d_1} \star \cdots \star \text{Asso}_{d_r}$$
Theorem (Pivot polytope of products of simplices)

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Example

(a) $\Pi_c(\square_d) \simeq \text{Perm}_d$
(b) $\Pi_c(\square_m \times \Delta_n) \simeq (m, n)$-multiplihedron
(c) $\Pi_c(\Delta_m \times \Delta_n) \simeq (m, n)$-constrainahedron
1) Is $\Pi_c(P)$ projection of a generalized permutahedron? 
$\rightarrow$ pivot fan sent inside $\text{Im}(\bar{\theta}) \cap B_{m'}$

2) For which $P$, $\Pi_c(P)$ is a generalized permutahedron? 
$\rightarrow$ a priori, only products of simplices, but no proof

3) When $\Pi_c(P)$ and $\Pi_c(Q)$ are not generalized permutahedra, then what happen to $\Pi_c(P \times Q)$? 
$\rightarrow$ not equivalent to $\Pi_c(P) \star \Pi_c(Q)$, but "embeds" in it

Thank you!
The polyhedral geometry of pivot rules and monotone paths, 2022.

On the geometric combinatorics of pivot rules.
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