Primitive elements for magmatic infinitesimal bialgebras

María Ronco
IMAFI. Universidad de Talca

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Joint work with D. Artenstein and A. González.

$\mathbb{K}$ a field, and $[n] := \{1, \ldots, n\}$, for $n \geq 1$.

Goal Study the bialgebra of special posets introduced by V. Pilaud and V. Pons in *Algebraic structures on integer posets* Proceedings of the 30th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2018), volume 80B, Art. 61, 12 pp, Hanover, United States. Séminaire Lotharingien de Combinatoire (2018).

SPoset_n the set of special posets with n elements $(P, \leq_P, \preceq_P)$ where P is a finite set, $\leq_P$ is a partial order and $\preceq_P$ is a total order on P.

Hopf algebra of finite special posets

$\mathbb{K}[\mathrm{SPoset}]$ is the graded vector space $\bigoplus_{n\geq 0} \mathbb{K}[\mathrm{SPoset}_n]$ with

1. **Product**

$$P * Q := \sum_{\substack{H \,\varepsilon\, SPoset_{n+m} \\ H|_{\{1,\dots,n\}}=P,\ H|_{\{1,\dots,n\}}=Q}} H,$$

   for $P \,\varepsilon\, \mathrm{SPoset}_n$ and $Q \,\varepsilon\, \mathrm{SPoset}_m$

2. **Coproduct** $\Delta(P) = \sum_{S\subseteq[n]} P|_S \otimes P|_{[n]\setminus S}$ where the sum is taken over all the subsets $S \subseteq [n]$ satisfying that $s <_P w$ for all $s \,\varepsilon\, S$ and $w \,\varepsilon\, [n] \setminus S$.

$(\mathbb{K}[\mathrm{SPoset}], \Delta)$ has too many primitive elements. In particular, any disconnected poset is primitive.

For $P \varepsilon \mathrm{SPoset}_n$ and $Q \varepsilon \mathrm{SPoset}_m$, consider the disjoint union $P \cup (Q+n)$. We have three posets structures on $P \cup (Q+n)$ satisfyng that $(P, \leq_P) \hookrightarrow (P \diamond Q, \leq_\diamond)$ and $(Q, \leq_Q) \hookrightarrow (P \diamond Q, \leq_\diamond)$ are morphism of posets for $\diamond \varepsilon \{\coprod, \uparrow, \downarrow\}$, and

(a) p and q are not comparable for $\leq_\coprod$ for $p \varepsilon P$ and $q \varepsilon Q$,

(b) $p \leq_\uparrow q$ whenever $p \varepsilon P$ and $q \varepsilon Q$,

(c) $q \leq_\downarrow p$ whenever $p \varepsilon P$ and $q \varepsilon Q$.

Dual graded Hopf algebra

Dual of the Pilaud-Pons Hopf algebra

1. **Coproduct $\overset{\circledast}{\Delta}$ dual of** $*$

$$\overset{\circledast}{\Delta}(P) := \sum_{i=0}^{n} P|_{\{1,\dots,i\}} \otimes P|_{\{i+1,\dots,n\}},$$

   for $P \,\varepsilon\, \mathrm{SPoset}_n$.

2. **Product $\circledast$ dual of Δ** $P \circledast Q := \sum_{\sigma\,\varepsilon\, Sh(n,m)} \sigma \cdot (P \uparrow Q)$, where $\sigma \cdot P$ has the same Hasse diagram than P and the total order on the nodes is obtained by replacing i by $\sigma(i)$.

Associated magmatic products

For $P \varepsilon \mathrm{SPoset}_n$ and $Q \varepsilon \mathrm{SPoset}_m$ and a pair of positive integers r, s define on $P \coprod Q = \{1, \ldots, n+m\}$ the partial orders

(a) $P \uparrow_r^s Q$ such that $(P \uparrow_r^s Q)|_{[n]} = P$, $(P \uparrow_r^s Q)|_{[m]+n} = Q + n$ and $i < j$ for $n - r < i \leq n$ and $n < j \leq n + s$,

(b) $P \downarrow_r^s Q$ such that $(P \downarrow_r^s Q)|_{[n]} = P$, $(P \downarrow_r^s Q)|_{[m]+n} = Q + n$ and $j < i$ for $n - r < i \leq n$ and $n < j \leq n + s$,

(c) $\tau_r(P, Q)$ such that $\tau_r(P, Q)|_{[n]} = P$, $\tau_r(P, Q)|_{[m]+n} = Q + n$ and $n - j < n + j + 1$ for $0 \leq j \leq r - 1$.

Remark The products $\coprod$, $\uparrow$ and $\downarrow$ are associative, but $\uparrow_r^s$, $\downarrow_r^s$ and τ_r are not associative.

But all of them satisfy the unital infinitesimal relation with $\overset{\circledast}{\Delta}$:

$$\overset{\circledast}{\Delta}(P \diamond Q) = \sum P_{(1)} \otimes (P_{(2)} \diamond Q) + \sum (P \diamond Q_{(1)}) \otimes Q_{(2)} - P \otimes Q.$$

Magmatic unital infinitesimal bialgebras

Definition

An *infinitesimal unital magmatic bialgebra* is a vector space A equipped with a unital product $\diamond$, an augmented coassociative coproduct $\overset{\circledast}{\Delta}$, satisfying

$$\overset{\circledast}{\Delta}(a \diamond b) = \sum a_{(1)} \otimes (a_{(2)} \diamond b) + \sum (a \diamond b_{(1)}) \otimes b_{(2)} - a \otimes b,$$

For a set S, an *S-infinitesimal unital magmatic bialgebra* is a vector space A equipped with a family $\{*_s\}_{s \,\varepsilon\, S}$ of unital products and an augmented coassociative coproduct $\overset{\circledast}{\Delta}$, satisfying that $(A, *_s, \overset{\circledast}{\Delta})$ is an infinitesimal unital magmatic bialgebra for all $s \,\varepsilon\, S$.

Free S-infinitesimal unital magmatic algebra

Let $\mathcal{T}_n$ be the set of planar rooted binary tree with n leaves.

Definition

Let $t \varepsilon \mathcal{T}_n$ and $w \varepsilon \mathcal{T}_m$.

1. The *wedge* of t and w is the tree $t \veebar w$ obtained by joining the roots of t and w.
2. The product $\curlywedge$ is
   2.1 $t \curlywedge | := t \veebar |$,
   2.2 $t \curlywedge w := (t \curlywedge w^l) \veebar w^r = w \circ_1 (t \veebar |)$,

   for $w = w^l \veebar w^r$.

$\curlywedge$ is associative.

Let $\mathbb{K}[\mathcal{T}^S] = \bigoplus_{n\geq 0} \mathbb{K}[\mathcal{T}_n^S]$ be the vector space spanned by the set of planar binary rooted trees with the internal nodes colored by the elements of S.
The elements of $\mathbb{K}[\mathcal{T}_n^S]$ are linear combintations of $(t, (s_1, \ldots, s_{n-1}))$.

$$(t, (s_1, \ldots, s_{n-1})) \underline{\vee}_s (w, (r_1, \ldots, r_{m-1})) = (t \underline{\vee} w, (s_1, \ldots, s_{n-1}, s, r_1, \ldots, r_{m-1})).$$

The free S-unital magmatic algebra over one generator is $(\mathbb{K}[\mathcal{T}^S, \underline{\vee}_s)$.

The coproduct $\overset{\circledast}{\Delta}$

As $(\mathbb{K}[\mathcal{T}^S], \underline{\vee}_s)$is free, there exist a unique coproduct $\overset{\circledast}{\Delta}$ on $\mathbb{K}[\mathcal{T}^S]$ satisfying that $(\mathbb{K}[\mathcal{T}^S], \underline{\vee}_s), \overset{\circledast}{\Delta})$ is a unital infinitesimal bialgebra for any $s \,\varepsilon\, S$.

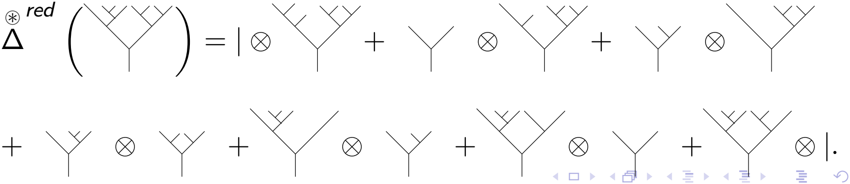
The linear map $\overset{\circledast}{\Delta} : \mathbb{K}[\mathbf{PBT}] \longrightarrow \mathbb{K}[\mathbf{PBT}] \otimes \mathbb{K}[\mathbf{PBT}]$ is defined recursively by

1. $\overset{\circledast}{\Delta}(1_{\mathbb{K}}) := 1_{\mathbb{K}} \otimes 1_{\mathbb{K}}$,
2. $\overset{\circledast}{\Delta}(|) := 1_{\mathbb{K}} \otimes | + | \otimes 1_{\mathbb{K}}$,
3. In general,
$$\overset{\circledast}{\Delta}\left(t \underline{\vee} w\right) := \sum t_{(1)} \otimes \left(t_{(2)} \underline{\vee} w\right) + \sum \left(t \underline{\vee} w_{(1)}\right) \otimes w_{(2)} - t \otimes w,$$
for any colored trees t and w, where $\overset{\circledast}{\Delta}(t) = \sum t_{(1)} \otimes t_{(2)}$ for any planar binary rooted tree.

Given a planar rooted binary tree t with n leaves, assume that the internal vertices of t are labelled, from left to right, by the elements of the set $\{1, \ldots, n-1\}$. The coproduct has the following expression:

$$\overset{\circledast}{\Delta}(t) = \sum_{i=0}^{n} t|_i \otimes t|^i,$$

where $t|_i$, respectively $t|^i$, is the tree obtained by eliminating the vertex labeled with i and keeping the tree on the left side, respectively keeping the tree on the right side, for $1 \leq i < n$.



An infinitesimal bialgebra in the sense of Joni-Rota

For any $t \in \mathcal{T}_n$, we define $\delta(t) \in \bigoplus_{i=0}^{n-1} \mathbb{K}[\mathcal{T}_{i+1}] \otimes \mathbb{K}[\mathcal{T}_{n-i}]$ as

$$\delta(t) := \sum_{i=0}^{n-1} t_{(1)}^{i+1} \otimes t_{(2)}^{i},$$

where $\overset{\circledast}{\Delta}(t) = \sum_{i=0}^{n} t_{(1)}^{i} \otimes t_{(2)}^{i}$ with $|t_{(1)}^{i}| = i$ and $|t_{(2)}^{i}| = n - i$.

Lemma

The coproducts $\overset{\circledast}{\Delta}$ and δ satisfy that

1. $(\mathbb{K}[\mathcal{T}], \curlywedge, \overset{\circledast}{\Delta})$ *is a unital infinitesimal bialgebra,*
2. δ *is a coderivation for the product $\underline{\vee}$, that is*

$$\delta \circ \underline{\vee} = (id \otimes \underline{\vee}) \circ (\delta \otimes id) + (\underline{\vee} \otimes id) \circ (id \otimes \delta),$$

3. $(\mathbb{K}[\mathcal{T}], \curlywedge, \delta)$ *is an infinitesimal bialgebra.*

Aguiar-Sottile formula for the coproduct

M. Aguiar and F. Sottile, *Structure of the Loday-Ronco Hopf algebra of trees* Journal of Algebra, Volume 295, Pages 473-511 (2006).

Definition

For any tree $t \varepsilon \mathcal{T}_n$,

$$M_t = \sum_{w \leq_T t} \mu(w;t) w,$$

where μ is the Möbius function of the Tamari order.

M. Aguiar and F. Sottile

$$\Delta(M_t) = \sum_{t_1/t_2 = t} M_{t_1} \otimes M_{t_2},$$

for the Hopf algebra of planar binary rooted trees (graded by the number of internal nodes) and for the Malvenuto-Reutenauer algebra, where $t_1/t_2 := t_2 \circ_1 t_1$.

Aguiar-Sottile formula for $\mathbb{K}[\mathcal{T}]$

Proposition The element $M_{\mathbf{1}_{T,n}}$ is given by the formulas

$$M_{\mathbf{1}_{T,n}} = | \veebar M_{\mathbf{1}_{T,n-1}} - | \not< M_{\mathbf{1}_{T,n-1}},$$

$$M_{\mathbf{1}_{T,n}} = \sum_{i=1}^{i-1} \mathbf{0}_{T,i} \veebar M_{\mathbf{1}_{T,n-i}},$$

for $n \geq 2$.

Proposition Let t be a planar rooted tree with n leaves, such that $t = \mathbf{1}_{T,r} \circ (t_1, \dots, t_{r-1}, |)$, with $r < n$ and $t_j \,\varepsilon\, \mathcal{T}_{n_j}$, for $1 \leq j < r$.

$$M_t = M_{\mathbf{1}_{T,r}} \circ (M_{t_1}, \dots, M_{t_{r-1}}, |).$$

Aguiar and Sottile's formula for magmatic bialgebras

Theorem

For any tree $t \in \mathbf{PBT}_n$*, we have that*

$$\overset{\circledast}{\Delta}(M_t) = \sum_{z_1 \curlywedge z_2 = t} M_{z_1} \otimes M_{z_2},$$

where $1_{\mathbb{K}} \curlywedge t = t = t \curlywedge 1_{\mathbb{K}}$ *for any planar rooted tree* t.

Primitive elements for S-magmatic algebras

Let S be a non-empty set and let $s_0 \varepsilon S$ be a fixed element. For any pair of colored trees $\underline{t}$ and $\underline{w}$ in $\bigcup_{n\geq 1} \mathbf{PBT}_n^S$, and any $s \neq s_0$, define

$$\underline{t} \vee_{\hat{s}} \underline{w} := \underline{t} \vee_s \underline{w} - \underline{t} \vee_{s_0} \underline{w}, \text{ for } s \neq s_0.$$

For $n \geq 1$, define the subset $\mathcal{I}_n$ of $\mathbb{K}[\mathbf{PBT}_n^S]$ as follows:

(i) $\mathcal{I}_1 := \{|\} = \mathbf{PBT}_1^S$,

(ii) $\mathcal{I}_2 := \{| \vee_{\hat{s}} |\}_{s \,\varepsilon\, S\setminus\{s_0\}}$,

(iii) for $n > 2, \mathcal{I}_n$ is the union of the subsets

- $\{(M_{|\vee t}, (s_1, \dots, s_{n-1})) \mid t \,\varepsilon\, \mathbf{PBT}_{n-1}, (s_1, \dots, s_{n-1}) \,\varepsilon\, S^{n-1}\}$,
- $\{\underline{t} \circ (\underline{w}_1, \dots . \underline{w}_r) \mid \underline{t}, \underline{w}_1, \dots, \underline{w}_r \,\varepsilon\, \bigcup_{i=1}^{n-1} \mathcal{I}_i, \text{ with } \sum_{j=1}^{r} |w_j| = n\}$,

and let $\mathbb{K}[\mathcal{I}_n]$ be the subspace of $\mathbb{K}[\mathbf{PBT}_n^S]$ spanned by $\mathcal{I}_n$.

Theorem

Let S be a non-empty set and $s_0 \varepsilon S$. Any element in $\mathbb{K}[\mathbf{PBT}_n^S]$ is a linear combination of elements of type

$$(\mathbf{0}_{T,r}, (s_0, \ldots, s_0)) \circ (\underline{t}_1, \ldots, \underline{t}_r),$$

where $\underline{t}_i \varepsilon \mathbb{K}[\mathcal{I}_{m_i}]$ for $1 \leq i \leq r$, with $\sum_{i=|}^{r} m_i = n$.

By R. Holtkamp, J:-L. Loday and M. R., *Coassociative magmatic bialgebras and the Fine numbers* Journal of Algebraic Combinatorics, Volume 28, Pages 97-114 (2008),
we get that any free S-magmatic infinitesimal bialgebra is isomorphic to the cotensor algebra of its primitive elements.
The theorem states that the subspace of primitive elements is the free algebra spanned by the elements $\underline{\vee}_{\hat{s}}$ and $M_{|\underline{\vee} t}$.

As $\mathbb{K}[\mathcal{T}^S]$ acts on any S-magmatic infinitesimal unital bialgebra, we get a family of primitive elements of the bialgebra of special posets.

Merci!