Relating Diagonals of the Permutahedra

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Let $P$ be a polytope in $\mathbb{R}^n$. In general, the set theoretic diagonal

$$\Delta : P \rightarrow P \times P$$

$$x \mapsto (x, x)$$

is not cellular.
A *cellular diagonal* of a polytope $P$ is a continuous map $P \rightarrow P \times P$ such that

1. its image is a union of dim $P$-faces of $P \times P$ (i.e. it is *cellular*),
2. it agrees with the thin diagonal on the vertices of $P$, and
3. it is homotopic to the thin diagonal, relative to the image of the vertices.
Cellular diagonals

Example

- Associahedron:
  - Saneblidze–Umble (2004),
  - Markl–Shnider (2006),
- Permutahedron:
  - Saneblidze–Umble (2004),
The Permutahedra

Definition

The \((n - 1)\)-dimensional permutahedra \(P_n\) is the convex hull of the points

\[(\sigma(1), \ldots, \sigma(n)) \in \mathbb{R}^n, \sigma \in S_n\]
Our Main Results

General enumeration results for **cellular** diagonals of the **permutahedra**

- Using hyperplane arrangements and a theorem of Zaslavsky.
- More explicit bijective formulae via Rainbow Trees/Forests

More general theory can be specialised to enumerate the diagonal!
Our Main Results

There exists an isomorphism $\Theta$ which decomposes each face $A_1|\ldots|A_k$ of the permutahedron $P_{|A_1|+\ldots+|A_k|-1}$ as a product $P_{|A_1|-1} \times \cdots \times P_{|A_k|-1}$.

Definition

A diagonal of the permutahedra $\triangle$ is operadic if for every face $A_1|\ldots|A_k$ of the permutahedron $P_{|A_1|+\ldots+|A_k|-1}$, the map $\Theta$ induces a topological cellular isomorphism

$$\triangle(A_1) \times \ldots \times \triangle(A_k) \cong \triangle(A_1|\ldots|A_k).$$

Theorem (BDO,MJV,GLA,VP,KS)

There are exactly two operadic diagonals of the permutahedra, which respect the weak Bruhat order on the vertices:

1. the LA diagonal of Laplante-Anfossi (2022), and
2. the SU diagonal of Saneblidze–Umble (2004).

They are isomorphic cellularly, and at the level of face lattices.
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A diagonal of the permutahedra $\triangle$ is \textit{operadic} if for every face $A_1|\ldots|A_k$ of the permutahedron $P_{|A_1|+\ldots+|A_k|-1}$, the map $\Theta$ induces a topological cellular isomorphism

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There exists an isomorphism $\Theta$ which decomposes each face $A_1|\ldots|A_k$ of the permutahedron $P|A_1|+\ldots+|A_k|−1$ as a product $P|A_1|−1 \times \cdots \times P|A_k|−1$.

Definition

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They are isomorphic cellurally, and at the level of face lattices.
The Goal for Today

Definition (Saneblidze–Umble, 2004)
The SU diagonal is given by the formula,

\[ \triangle^{SU}([n]) = \bigcup (\sigma, \tau) \bigcup M, N R_M(\sigma) \times L_N(\tau) \]

where the unions are taken over all strong complementary partitions \((\sigma, \tau)\) of \([n]\), and over all admissible sequences of shifts \(M, N\).

Definition (Laplante-Anfossi, 2022)
The LA diagonal is given by \(\vec{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n\), which satisfy

\[ \sum_{i \in I} v_i > \sum_{i \in J} v_j \quad \forall (I, J) \in LA(n) \]
The Diagonals

Let \( O(n) := \{(I, J) \mid I, J \subset [n], |I| = |J|, I \cap J = \emptyset\} \)

**Definition**

We define LA\((n)\) and SU\((n)\) as subsets of \( O(n) \),

- LA\((n)\) := \( \{(I, J) \in O(n) \mid \min(I \cup J) = \min I\} \), and by
- SU\((n)\) := \( \{(I, J) \in O(n) \mid \max(I \cup J) = \max J\} \).

**Example**

Underlined in LA, and overlined in SU,

\[
O(2) = \{(1,2), (2,1)\}
\]

\[
O(3) \ni (1,3), (2,3), (2,1), (3,2)
\]

\[
O(4) \ni (1,2), (3,2), (14,23), (23,14), (13,24)
\]
Geometric Formulae

Definition (Laplante-Anfossi, 2022)

The LA diagonal is given by \( \vec{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n \), satisfying

\[
\sum_{i \in I} v_i > \sum_{i \in J} v_j, \quad \forall (I, J) \in \text{LA}(n)
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## Geometric Formulae

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### Definition

The 'SU Geometric diagonal' is given by $\vec{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$, satisfying

$$\sum_{i \in I} v_i > \sum_{j \in J} v_j, \quad \forall (I, J) \in \text{SU}(n)$$
**Geometric Formulae**

**Definition (Laplante-Anfossi, 2022)**

The LA diagonal is given by \( \vec{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n \), satisfying

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\sum_{i \in I} v_i > \sum_{i \in J} v_j, \quad \forall (I, J) \in \text{LA}(n)
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**Definition**

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\sum_{i \in I} v_i > \sum_{i \in J} v_j, \quad \forall (I, J) \in \text{SU}(n)
\]

**Theorem (BDO,MJV,GLA,VP,KS)**

*This geometric definition of \( \Delta_{SU} \) recovers the original definition of \( \Delta_{SU} \).*
A Geometric Formula

**Definition (Laplante-Anfossi, 2022)**

The LA diagonal is given by \( \vec{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n \), satisfying

\[
\sum_{i \in I} v_i > \sum_{i \in J} v_j , \; \forall (I, J) \in \text{LA}(n)
\]

**Theorem (Laplante-Anfossi, 2022)**

For a pair \((\sigma, \tau)\) of ordered partitions of \([n]\), we have

\[
(\sigma, \tau) \in \Delta_{\text{LA}} \iff \forall (I, J) \in \text{LA}(\sigma, \tau), \exists k \in [n], |\sigma[k] \cap I| > |\sigma[k] \cap J| \text{ or } \\
\exists l \in [n], |\tau[l] \cap I| < |\tau[l] \cap J| \\
\iff \forall (I, J) \in \text{LA}(n), \exists k \in [n], |\sigma[k] \cap I| > |\sigma[k] \cap J| \text{ or } \\
\exists l \in [n], |\tau[l] \cap I| < |\tau[l] \cap J| .
\]
A Combinatorial Interpretation

Definition
A \( n \)-partition tree is a pair \((\sigma, \tau)\) of set partitions of \([n]\) whose intersection graph is a bipartite tree.

Example
An example and counter example,

\[
\begin{align*}
13 &| 24 | 57 | 6 \times 17 | 2 | 3 | 456 & & 13 &| 24 | 57 | 6 \times 1 | 27 | 3 | 456 \\
\begin{array}{c}
13 \\
24 \\
57 \\
6
\end{array} & \begin{array}{c}
17 \\
2 \\
3 \\
456
\end{array} & \begin{array}{c}
13 \\
24 \\
57 \\
6
\end{array} & \begin{array}{c}
1 \\
27 \\
3 \\
456
\end{array}
\end{align*}
\]
Proposition (BDO,MJV,GLA,VP,KS)

Let \((\sigma, \tau)\) be a pair of ordered partitions of \([n]\) forming an \(n\)-partition tree. If for all pairs of adjacent blocks, the directed path between them traverses the maximal path element right to left, then \((\sigma, \tau) \in \bigtriangleup^{SU}\).

\(\sigma\) is Good:

\(\tau\) is Bad:
Re-orienting

Proposition (BDO, MJV, GLA, VP, KS)

*Every $n$-partition tree can be uniquely oriented into an element of $\triangle^{SU}$.*

\[
\begin{align*}
13 | 24 | 57 | 6 & \times 17 | 2 | 3 | 456 & 13 | 24 | 57 | 6 & \times 3 | 17 | 456 | 2 \\
13 & 17 \\
24 & 2 \\
57 & 3 \\
6 & 456 & 13 & 3 \\
24 & 17 \\
57 & 456 \\
6 & 2 & \in \triangle^{SU}
\end{align*}
\]
Re-orienting

**Proposition (BDO,MJV,GLA,VP,KS)**

*Every $n$-partition tree can be uniquely oriented into an element of $\triangle^{SU}$.***

\[13|24|57|6 \times 17|2|3|456\]
\[\rightarrow\]
\[13|24|57|6 \times 3|17|456|2\]
\[\in \triangle^{SU}\]
Geometry Informs Combinatorics

$$(\sigma, \tau) \in \triangle^{SU} \iff \forall (I, J) \in SU(\sigma, \tau), \exists k \in [n], |\sigma[k] \cap I| > |\sigma[k] \cap J| \text{ or } \exists l \in [n], |\tau[l] \cap I| < |\tau[l] \cap J|$$

$SU(\sigma, \tau) = \{(I, J) \text{ encoded in paths between adj. blocks}\}$

Existential Statement $\cong$ Maximal path element traversed right to left

$$(I, J) = (\{1, 5\}, \{4, 7\})$$
The Diagonal Via Shifts

Definition (Saneblidze–Umble, 2004)

The SU diagonal is given by the formula,

\[ \Delta^{SU}([n]) = \bigcup_{(\sigma, \tau)} \bigcup_{M,N} R_{M}(\sigma) \times L_{N}(\tau) \]

where the unions are taken over all strong complementary partitions \((\sigma, \tau)\) of \([n]\), and over all admissible sequences of shifts \(M, N\).
Strong Complementary Partitions

**Definition**

Given a permutation $\nu$, we define its strong complementary pair $(\sigma, \tau)$ by,

- $\sigma$ is obtained by merging all decreasing sequences of $\nu$
- $\tau$ is obtained by merging all increasing sequences of $\nu$

**Proposition**

The maximal path elements of SCPs are always traversed right to left.
Strong Complementary Partitions

Definition
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Proposition
The maximal path elements of SCPs are always traversed right to left.
Let $\sigma = \sigma_1 \ldots |\sigma_k$ be an ordered partition, and let $M_i \subsetneq \sigma_i$ be a non-empty subset of the block $\sigma_i$. We define the right/left shift operators:

$$R_{M_i}(\sigma) := \sigma_1 \ldots |\sigma_i \setminus M_i|\sigma_{i+1} \cup M_i| \ldots |\sigma_k$$

$$L_{M_i}(\sigma) := \sigma_1 \ldots |\sigma_{i-1} \cup M_i|\sigma_i \setminus M_i| \ldots |\sigma_k .$$
Admissible Shifts

Definition

Let $\sigma = \sigma_1|\ldots|\sigma_k$ be an ordered partition

- A right shift is admissible if $\min \sigma_i \not\in M_i$, and $\min M_i > \max \sigma_{i+1}$.

Dually,

- A left shift is admissible if $\min \sigma_i \not\in M_i$, and $\min M_i > \max \sigma_{i-1}$.
Admissible Shifts

**Definition**

Let $\sigma = \sigma_1 \ldots \sigma_k$ be an ordered partition

- A right shift is admissible if $\min \sigma_i \not\in M_i$, and $\min M_i > \max \sigma_{i+1}$.
- A sequence of right shifts $\mathbf{M} = (M_{i_1}, \ldots, M_{i_p})$, is admissible if $i_1 < \ldots < i_p < k$, and each sequential shift is admissible.

Dually,

- A left shift is admissible if $\min \sigma_i \not\in M_i$, and $\min M_i > \max \sigma_{i-1}$.
- A sequence of left shifts $\mathbf{M} = (M_{i_1}, \ldots, M_{i_p})$, is admissible if $i_1 > \ldots > i_p > 1$, and each sequential shift is admissible.
The Diagonal Via Shifts

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The SU diagonal is given by the formula,

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Proposition (BDO,MJV,GLA,VP,KS)

Let \((\sigma, \tau)\) be a pair of ordered partitions of \([n]\) forming an \(n\)-partition tree. If for all pairs of adjacent blocks, the directed path between them traverses the \textbf{maximal} path element \textbf{right to left}, then \((\sigma, \tau) \in Geo. \ \triangle^{SU}\).

Show all elements of shift \(\triangle^{SU}\) also satisfy the path condition.

1. We know strong complementary partitions meet the path condition,
2. We show admissible sequences of shifts conserve the path condition,

Consequently, Shift \(\triangle^{SU} \subseteq Geometric \ \triangle^{SU}\).
Geometric $\Delta^\text{SU} \subseteq \text{Shift} \Delta^\text{SU}$.

Conversely we need,

**Lemma**

Let $(\sigma, \tau)$ be a pair of ordered partitions of $[n]$ forming an $n$-partition tree. If for all pairs of adjacent blocks, the directed path between them traverses the **maximal** path element **right to left**, then it is either a strong complementary pair, or generated by shifts.

Idea: For anything that is not a strong complementary partition we can identify an inverse shift operator, e.g.

$$
\begin{array}{c}
13 & \rightarrow & 3 \\
24 & \rightarrow & 17 \\
57 & \rightarrow & 456 \\
6 & \rightarrow & 2
\end{array}
$$

$$
R_7^{-1} \times \text{id}
$$

$$
\begin{array}{c}
13 & \rightarrow & 3 \\
247 & \rightarrow & 17 \\
5 & \rightarrow & 456 \\
6 & \rightarrow & 2
\end{array}
$$

$\rightarrow \cdots$
This geometric definition of $\triangle^{\text{SU}}$ recovers the original definition of $\triangle^{\text{SU}}$. 

**Theorem (BDO,MJV,GLA,VP,KS)**
This geometric definition of $\triangle^{SU}$ recovers the original definition of $\triangle^{SU}$.

Consequently, have many different encodings of the LA and SU diagonals.

- Geometric formulae
- Min/max path formulae
- Shift formulae
- Cubical formulae
- Matrix formulae
Theorem (BDO,MJV,GLA,VP,KS)

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\[ \triangle_{SU} \ni 13 \quad 3 \quad 24 \quad 17 \quad t \quad 3 \quad 17 \quad 24 \quad r \times r \quad 5 \quad 57 \quad 13 \quad 234 \quad 6 \quad \in \triangle_{LA} \]

**Proposition (BDO,MJV,GLA,VP,KS)**

Let \((\sigma, \tau)\) be a pair of ordered partitions of \([n]\) forming an \(n\)-partition tree. If for all pairs of adjacent blocks, the directed path between them traverses

1. the maximal path element right to left, then \((\sigma, \tau) \in \triangle_{SU}\).
2. the minimal path element left to right, then \((\sigma, \tau) \in \triangle_{LA}\).