Some algebraic and combinatorial problems from non-commutative probability

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Sequences with a combinatorial/probabilistic flavor

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Moment problem: $(a_n)_n$ is the sequence of moments of some measure if and only if the Hankel matrices associated to the sequence are positive definite.

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be the *formal Laplace transform*. We can write this series as

$$\mathcal{F}(z) = e^{K(z)},$$

where

$$K(z) = \sum_{n \geq 1} \kappa_n \frac{z^n}{n!}$$

is the cumulant generating function.

By the *exponential formula*, since

$$\mathcal{F}(z) = e^{\mathsf{K}(z)},$$

then we have

$$m_\pi = \sum_{\pi \leq \tau} \kappa_\tau,$$

where $m_\pi = m_{|B_1|} m_{|B_2|} \cdots m_{|B_k|}$ and $\kappa_\pi = \kappa_{|B_1|} \kappa_{|B_2|} \cdots \kappa_{|B_k|}$ if $\pi = \{B_1, B_2, \ldots, B_k\}$.

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Here, $\leq$ corresponds to the poset of partitions $\Pi(n)$ of the set $[n] := \{1, 2, \ldots, n\}$ with the refinement order. The minimal element is the partition $\{[n]\}$. Hence,

$$m_n = \sum_{\pi \in \Pi(n)} \kappa_\pi \qquad \text{and} \qquad \kappa_n = \sum_{\pi \in \Pi(n)} \mu(\widehat{0}, \pi) m_\pi.$$

If $f(n) := a_n$ for all $n \geq 0$, consider the following sequences associated to f:

- the (classical) cumulant sequence $(k_n(f)_n)_{n\geq 0}$:

$$k_n(f) := \sum_{\pi \in \Pi(n)} \mu(\widehat{0}, \pi) f(\pi);$$

- the free cumulant sequence $(c_n(f))_{n\geq 0}$:

$$c_n(f) := \sum_{\pi \in NC(n)} \mu(\widehat{0}, \pi) f(\pi);$$

- the boolean sequence $(b_n(f))_{n\geq 0}$:

$$b(f)_n := \sum_{\pi \in NC_{int}(n)} \mu(\widehat{0}, \pi) f(\pi)$$

For example,

$$f(\{\{3,8,9\},\{1,2\},\{6\},\{4,6,7\}\}) = a_{|\{3,8,9\}|} \cdot a_{|\{1,2\}|} \cdot a_{|\{6\}|} \cdot a_{|\{4,6,7\}|} = a_1 a_2 a_3^2.$$

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Goal: understand these new sequences when $f(n)$ arises as *dimensions of combinatorial spaces*.

Species

Species

André Joyal, Alain Connes, Olivia Caramello and Laurent Lafforgue, IHES (2015)

The theory of *combinatorial species* was introduced by André Joyal in 1980. Species can be seen as a *categorification* of generating functions. It provides a categorical foundation for enumerative combinatorics.

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A **set-species** is a functor

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The **Cauchy product** of two species p and q is given by

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The category of species is symmetric monoidal.
We can speak of monoids, comonoids, ..., in species.

$$\mathrm{h}[S] \otimes \mathrm{h}[T] \xrightarrow{\mu_{S,T}} \mathrm{h}[I] \qquad \mathrm{h}[I] \xrightarrow{\Delta_{S,T}} \mathrm{h}[S] \otimes \mathrm{h}[T].$$

Examples of species

- Species E of **sets**:

$$E[I] := \mathbb{K}\{*_I\}.$$

- Species E_n of n-**sets**:

$$E_n[I] := \begin{cases} \mathbb{K}\{*_I\}, & \text{if } |I| = n; \\ (0), & \text{if } |I| \neq n. \end{cases}$$

- Species $X := E_1$ of sets of one element.
- Species Π of **partitions**.
- Species L of **linear orders**.
- Species G of **graphs**:

$$G[I] := \mathbb{K}\{\text{ finite graphs with vertices in } I\,\}.$$

Examples of species

- Species B of **binary trees**.
- Species $\mathfrak{S}$ of **permutations**.
- Species Braid of **braid hyperplane arrangements**.

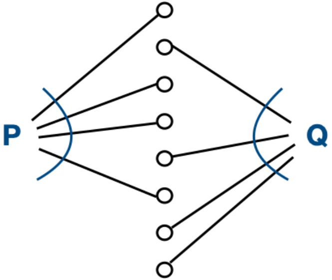
Operations on species

- **Sum of species**

$$(p + q)[I] := p[I] \oplus q[I].$$

- **Product of species** (Cauchy product)

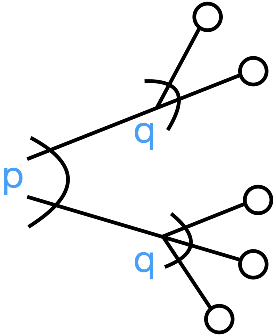
$$(p \cdot q)[I] := \bigoplus_{I = S \sqcup T} p[S] \otimes q[T].$$



Operations on species

- **Composition of species**

$$(\mathsf{p} \circ \mathsf{q})[I] := \bigoplus_{\pi \in \Pi[I]} \mathsf{p}[\pi] \otimes \bigotimes_{B \in \pi} \mathsf{q}[B].$$



Generating function of a species

To every species p it is associated its **exponential generating function**:

$$p(x) := \sum_{n \geq 0} \dim_{\mathbb{K}} p[n] \frac{x^n}{n!}.$$

We have:

$$(p + q)(x) = p(x) + q(x),$$

$$(p \cdot q)(x) = p(x) \cdot q(x),$$

$$(p \circ q)(x) = p(x) \circ q(x).$$

For the last identity, $q[\emptyset] := (0)$.

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which implies:

$$\mathsf{B}(x) = x + \mathsf{B}(x)^2/2.$$

Therefore,

$$\mathsf{B}(x) = 1 - \sqrt{1-2x} = \sum_{n \geq 1} 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3) \frac{x^n}{n!}.$$

Marcelo Aguiar, Swapneel Mahajan

Cumulants from species

Based on Aguiar, M., Mahajan, S. (2013). *Hopf monoids in the category of species*, Hopf algebras and tensor categories, 585, 17-124.

Cumulants from Hopf monoids

Let I be a finite set.
Let $\pi \vdash I$ be a partition of I. For a species h, consider

$$h(\pi) := \bigotimes_{B \in \pi} h[B].$$

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The cumulants of h are the integers $k_\pi(\mathrm{h})$ defined by

$$k_\pi(\mathrm{h}) = \sum_{\tau :\, \tau \geq \pi} \mu(\pi, \tau)\, \dim_{\Bbbk} \mathrm{h}(\tau).$$

Cumulants from Hopf monoids (Aguiar-Mahajan)

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$$k_n(h) = \sum_{\pi \vdash I} \mu(\{I\}, \pi) \dim_{\Bbbk} h(\pi).$$

Hopf monoid	Moments	Cumulants	Distribution
L *linear orders*	$n!$	$(n-1)!$	Exponential of par. 1
E *sets*	1	$\delta_{n,1}$	Dirac measure $\delta = 1$
Π *partitions*	Bell_n	1	Poisson of par. 1
Σ *ordered partitions*	$\mathrm{OrdBell}_n$	$\sum_{k \geq 1} \frac{k^n}{2^k}$	Geometric of par. 1

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Proposition (Aguiar-Mahajan)

For any finite-dimensional cocommutative connected bimonoid h*, the dimension of its primitive part is*

$$\dim_{\Bbbk} \mathcal{P}(\mathsf{h})[I] = k_{|I|}(\mathsf{h}).$$

Free and boolean cumulants of h

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Question: are these integers non-negative? What conditions on h?

Generating functions

Given a species h, the *ordinary*, *type* and *exponential* generating functions of h are, respectively,

$$\mathsf{Exp}_{\mathsf{h}}(z) := \sum_{n \geq 0} \dim_{\Bbbk} \mathsf{h}[n] \, \frac{z^n}{n!}, \quad \mathsf{T}_{\mathsf{h}}(z) = \sum_{n \geq 0} \dim_{\Bbbk} \mathsf{h}[n]_{\mathfrak{S}_n} \, z^n$$

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If h is connected and finite-dimensional, then $1 - 1/\mathrm{O}_{\mathsf{h}}(z) \in \mathbb{N}[[z]]$.

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No assumptions on cocommutativity.

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No assumptions on cocommutativity. What about $\mathrm{c}_n(\mathsf{h})$?

Recall that the composition of species is given by

$$(\mathsf{p} \circ \mathsf{q})[I] := \bigoplus_{\pi \in \Pi[I]} \mathsf{p}[\pi] \otimes \bigotimes_{B \in \pi} \mathsf{q}[B].$$

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Let $k \geq 0$. Given a species p, the k-**divided power of** p is the species $\gamma_k(p)$ of all k-assemblies of p-structures:

$$\gamma_k(p)[I] := \bigoplus_{\pi \in \Pi_k[I]} p(\pi).$$

Then,

$$(p \circ q)[I] = \sum_{k \geq 0} p[k] \otimes \gamma_k(p)[I].$$

Non-crossing composition

Given a *linear species* r, consider the new species $\overrightarrow{\gamma}_k(\mathsf{r})$ given by

$$\overrightarrow{\gamma}_k(\mathsf{r})[I] := \bigoplus_{\pi \in \mathsf{NC}_k(I)} \mathsf{r}(\pi).$$

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For every species p, we define the non-crossing k-divided power of p as

$$\gamma_{\mathrm{k}}^{\mathsf{NC}}(\mathsf{p}) := \mathsf{L} \times \overrightarrow{\gamma}_{\mathrm{k}}(\mathsf{p}).$$

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The non-crossing composition of two species p and q is defined as

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$$(p \circ_{NC} q)[I] := \sum_{k \geq 0} p[k] \otimes \gamma_k^{NC}(q)[I].$$

For example, $E \circ E_+$ is the *species of partitions*, while $E \circ_{NC} E_+$ is the *species of non-crossing partitions*.

Non-crossing composition

Given two ordinary generating functions

$$A(z) = \sum_{n \geq 0} a_n z^n \qquad , \qquad B(z) = \sum_{n \geq 0} b_n z^n,$$

with $b_0 = 0$, the non-crossing composition of series is

$$(A \circ_{NC} B)(z) := \sum_{n \geq 0} \left(\sum_{k \geq 0} k! a_k \sum_{\pi \in NC_k(n)} \pi! \, b_\pi \right) z^n.$$

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Proposition (V. - 2023)

Given species p *and* q, *with* $q[\emptyset] = 0$, *we have*

$$\mathrm{Exp}_{p \circ_{NC} q}(z) = (\mathrm{Exp}_p \circ_{NC} \mathrm{Exp}_q)(z).$$

Non-crossing composition

The *exponential formula* in combinatorics can be expressed in term of species as

$$\begin{aligned}\mathsf{Exp}_{\mathrm{E}\circ\mathsf{p}}(z) &= (\mathsf{Exp}_{\mathsf{E}} \circ \mathsf{Exp}_{\mathsf{p}})(z)\\ &= e^{\mathsf{Exp}_{\mathsf{p}}(z)}.\end{aligned}$$

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Corollary (V. 2023)

There is a non-crossing exponential formula as follows:

$$\mathsf{Exp}_{\mathrm{E}\circ_{\mathsf{NC}}\mathsf{p}}(z) = (\mathsf{Exp}_{\mathsf{E}} \circ_{\mathsf{NC}} \mathsf{Exp}_{\mathsf{p}})(z).$$

Let h be a species. Recall that we have sequences of type of cumulants associated to h:

$$k_n(\mathsf{h}) = \sum_{\pi \in \Pi(n)} \mu(\{I\}, \pi) \dim_{\mathbb{k}} \mathsf{h}(\pi), \; c_n(\mathsf{h}) = \sum_{\pi \in \mathsf{NC}(n)} \mu(\{I\}, \pi) \dim_{\mathbb{k}} \mathsf{h}(\pi).$$

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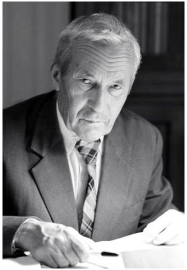
Corollary (V. - 2023)

Let p *be a positive species.*

- *if* $h = E \circ p$, *then,* $k_{|I|}(h) = \dim_{\Bbbk} p[I]$;
- *if* $h = E \circ_{NC} p$, *then,* $c_{|I|}(h) = \dim_{\Bbbk} p[I]$;
- *if* $h = E \diamond p$, *then,* $b_{|I|}(h) = \dim_{\Bbbk} p[I]$.

Non-commutative probability

Classical probability space



Andrey Kolmogorov

A **probability space** (Kolmogorov, 1930's) is given by the following data:

- a set Ω (sample space),
- a collection $\mathcal{F}$ (event space),
- $\mathbb{P} : \mathcal{F} \to [0, 1]$ (probability function),

satisfying several axioms.

Expectation: for every random variable $X \in L^{\infty}(\Omega, \mathcal{F}, \mathbb{P})$, let

$$\mathbb{E}[X] := \int_{\Omega} X(\omega)\, d\mathbb{P}(\omega).$$

Intuition: replace $(L^{\infty}(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$ by a more general pair $(\mathcal{A}, \varphi)$.

Non-commutative probability space

A **non-commutative probability space** is a pair $(\mathcal{A}, \varphi)$ such that

- $\mathcal{A}$ is a unital associative algebra over $\mathbb{C}$;
- $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\varphi(1_{\mathcal{A}}) = 1$.

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Examples: $(L^\infty(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$, $\left(\mathrm{Mat}_n(\mathbb{C}), \frac{1}{n}\mathrm{Tr}\right)$, $(\mathrm{Mat}_n(\Omega), \varphi)$.

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$$\varphi(a) := \int_\Omega \mathrm{tr}(a(\omega))\, d\mathbb{P}(\omega)$$

- The field of *Free Probability* was created by Dan Voiculescu in the 1980s.
- Philosophy: investigate the notion of “freeness" in analogy to the concept of “independence” from (classical) probability theory.
- A combinatorial theory of freeness was developed by Nica and Speicher in the 1990s.
- Voiculescu discovered freeness also asymptotically for many kinds of random matrices (1991).

Dan Voiculescu , 2015

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In a (classical) probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the notion of independence between two random variables $X, Y : \Omega \to \mathbb{C}$ implies

$$\mathbb{E}(X^m Y^n) = \mathbb{E}(X^m)\mathbb{E}(Y^n).$$

Non-commutative independence

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Consider $\{\mathcal{A}_i\}_{i \in I}$ unital subalgebras of $\mathcal{A}$. Let $a_1 \in \mathcal{A}_{i_1}, \ldots, a_n \in \mathcal{A}_{i_n}$ such that $i_j \neq i_{j+1}$.

The family $\{\mathcal{A}_i\}_{i \in I}$ is

- **freely independent** if

  $$\varphi(a_1 \cdots a_n) = 0,$$

  when $\varphi(a_j) = 0$, for all $1 \leq j \leq n$;
- **boolean independent** if

  $$\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n);$$

Other notions: *monotone independence*, *conditional monotone*, ...

Moment to cumulant relations in $(\mathcal{A}, \varphi)$

Consider the multilinear functionals

$\{r_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ (Free cumulants) , $\{b_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ (Boolean cumulants) , $\{h_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ (Monotone cumulants)

defined by

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} r_\pi(a_1, \ldots, a_n),$$

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC_{\mathrm{Int}}(n)} b_\pi(a_1, \ldots, a_n),$$

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \frac{1}{\tau(\pi)!} h_\pi(a_1, \ldots, a_n).$$

Algebraic approach to cumulants (Ebrahimi-Fard, Patras)

- $(\mathcal{A}, \varphi)$ non-commutative probability space.

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- The linear form φ is extended to $T_+(\mathcal{A})$ by defining to all words $u = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$

$$\varphi(a_1 a_2 \cdots a_n) := \varphi(a_1 \cdot_{\mathcal{A}} a_2 \cdot_{\mathcal{A}} \cdots \cdot_{\mathcal{A}} a_n).$$

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This is the **multivariate moment** of u.
The map φ is then extended multiplicatively to a map $\Phi : T(T_+(\mathcal{A})) \to \mathbb{K}$ with $\Phi(\mathbf{1}) := 1$ and

$$\Phi(u_1 | \cdots | u_k) := \varphi(u_1) \cdots \varphi(u_k).$$

Cumulants as infinitesimal characters

Proposition (Ebrahimi-Fard, Patras -2015)

Let $\rho, \kappa, \beta \in \mathfrak{g}(\mathcal{A})$ *the infinitesimal characters solving*

$$\Phi = \exp_*(\rho),$$

$$\Phi = \epsilon + \kappa \prec \Phi$$

and

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Then, ρ, κ, β *correspond to the* ***monotone cumulants, free cumulants and boolean cumulants****, respectively.*

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For any word $u = a_1 \cdots a_n \in \mathcal{A}^{\otimes n}$, we have

$$h_n(a_1, \dots, a_n) = \rho(u), r_n(a_1, \dots, a_n) = \kappa(u), b_n(a_1, \dots, a_n) = \beta(u).$$

Series on species

From species to vector spaces I

There are functors

$$\mathcal{K}, \overline{\mathcal{K}}, \mathcal{K}^\vee, \overline{\mathcal{K}} : \text{Hopf monoids in species} \longrightarrow \mathbb{N}\text{-graded Hopf algebras.}$$

$$\mathcal{K}(\mathsf{h}) = \mathcal{K}^\vee(\mathsf{h}) := \bigoplus_{n \geq 0} \mathsf{h}[n]$$

$$\overline{\mathcal{K}}(\mathsf{h}) := \bigoplus_{n \geq 0} \mathsf{h}[n]_{\mathfrak{S}_n} \quad , \quad \overline{\mathcal{K}}^\vee(\mathsf{h}) := \bigoplus_{n \geq 0} \mathsf{h}[n]^{\mathfrak{S}_n}$$

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Patras-Schocker-Reutenauer:

$\mathcal{K}(\mathrm{h})$: cosymmetrized bialgebra

$\mathcal{K}^{\vee}(\mathrm{h})$: symmetrized bialgebra

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- $\mathcal{K}(\mathrm{h}) \cong \overline{\mathcal{K}}(\mathrm{L} \times \mathrm{h})$.
- If h is finite-dimensional, then $\overline{\mathcal{K}}(\mathrm{h}^*) \cong \overline{\mathcal{K}}(\mathrm{h})^*$.
- If h is cocommutative, then so are $\mathcal{K}(\mathrm{h})$ and $\overline{\mathcal{K}}(\mathrm{h})$.
- If h is commutative, so is $\overline{\mathcal{K}}(\mathrm{h})$.

From species to vector spaces II

Let p be a species.

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A **series** s of p is a collection of elements

$$s_I \in \mathrm{p}[I],$$

one for each finite set I, such that

$$\mathrm{p}[\sigma](s_I) = s_J,$$

for each bijection $\sigma : I \to J$.

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The space $\mathscr{S}(\mathsf{p})$ of all series of p is a vector space:

$$(s + t)_I = s_I + t_I \quad , \quad (\lambda \cdot s)_I := \lambda\, s_I,$$

for $s, t \in \mathscr{S}(\mathsf{p})$ and $\lambda \in \mathbb{K}$.

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Let E be the exponential map. A series s of p corresponds to the morphism of species

$$\mathsf{E} \longrightarrow \mathsf{p}$$

$$*_I \longmapsto s_I,$$

so $\mathscr{S}(\mathsf{p}) \cong \mathsf{Hom}_{\mathsf{Sp}}(\mathsf{E}, \mathsf{p})$.

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for each bijection $\sigma : I \to J$.

Property (2) implies that each $s_{[n]}$ is an $\mathfrak{S}_n$-invariant element of $\mathrm{p}[n]$. In fact,

$$\mathscr{S}(\mathrm{p}) \cong \prod_{n \geq 0} \mathrm{p}[n]^{\mathfrak{S}_n}$$

$$s \mapsto (s_{[n]})_{n \geq 0}.$$

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$$s_{\mathrm{I}} \in \mathsf{p}[\mathrm{I}],$$

one for each finite set I, such that

$$\mathsf{p}[\sigma](s_{\mathrm{I}}) = s_{\mathrm{J}}, \tag{2}$$

for each bijection $\sigma : \mathrm{I} \to \mathrm{J}$.

There is a functor

$$\mathscr{S} : \mathsf{Sp} \to \mathsf{Vec}.$$

The functor $\mathscr{S}$ is *braided lax monoidal*: it preserves monoids, commutative monoids, Lie monoids ...

Decorated series

Let V be a vector space.

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Let V be a vector space. Recall that a series of p corresponds to a morphism of species $\mathsf{E} \longrightarrow \mathsf{p}$.

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A V-**decorated series**, or **decorated series**, is a morphism of species

$$\mathsf{E}_V \longrightarrow \mathsf{p},$$

where E_V is the **exponential decorated exponential** given by

$$\mathsf{E}_V[I] := \mathbb{K}\{f : I \to V\}.$$

Let $\mathscr{S}_V(\mathsf{p})$ be the space of decorated series.

Decorated series

A series s in $\mathscr{S}_V(\mathsf{p})$ is a collection of elements

$$s_{I,f} \in \mathsf{p}[I],$$

one for each finite set I and for each map $f : I \to V$, such that

$$\mathsf{p}[\sigma](s_{I,f}) = s_{J,f\circ\sigma^{-1}},$$

for each bijection $\sigma : I \to J$.

Cumulants from decorated series (V., 2023)

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space.

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Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space.

Consider the *ripping and sewing* Hopf monoid P. As a species, $\mathsf{P} = \mathsf{L} \circ \mathsf{L}_+$.

Define $\Phi \in \mathscr{S}_{\mathcal{A}}(\mathsf{P}^*)$ as follows: if I is a finite set and $f : I \to \mathcal{A}$, let

$$\Phi_{I,f} \in \mathsf{P}^*[I]$$

given by

$$\Phi_{I,f}(w_1 w_2 \cdots w_n) := \varphi(w_1) \cdots \varphi(w_n),$$

where for each $w_k = x_1^k \cdots x_r^k \in \mathsf{L}_+[I_k]$,

$$\varphi(w) := (\varphi \circ f)(x_1^k) \cdots (\varphi \circ f)(x_r^k).$$

Cumulants from decorated series

Proposition (V. - 2023)

Let $(\mathcal{A}, \varphi)$ *be a non-commutative probability space. For every species p, consider the space* $\mathrm{C}_{\mathcal{A}}(p) := \mathscr{S}_{\mathcal{A}}((L \circ p_{+})^{*})$.

- *Classical cumulants are obtained from* $p = X$
- *Non-commutative cumulants are obtained from* $p = L$

Problem : structure on p giving a more general ripping and sewing coproduct on the *free monoid* $\mathrm{L} \circ \mathrm{p}_{+}$?

(In progress: structure of *hereditary species* on p)

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- More general notion (replace the algebra $\mathcal{A}$ by a monoid in species h):

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- More general notion (replace the algebra $\mathcal{A}$ by a monoid in species h):

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Particular case: $\mathsf{p} := \mathsf{X}$, (h, φ) a connected bimonoid with

$$\varphi_I(x) := \dim_{\mathbb{K}} h[I],$$

for all $x \in h[I]$.

Work in progress

- An algebraic model for several notions of non-commutative independences was presented by Ebrahimi-Fard and Patras. It involves infinitesimal characters on a certain Hopf algebra.
- Understanding this approach in terms of species and algebraic structures in the monoidal category of species (monoids, comonoids, lie monoids, bimonoids) might give a better insight of the combinatorics behind moment-to-cumulant formulae.
- Universality of $\mathrm{E} \circ_{\mathrm{NC}} \mathrm{p}$ (analogue to the *free and cofree* monoid in species).
- Operadic notion using non-crossing composition (rigid and classic species).
- What's next?

Geometrical notion of independence(s)?

Polytope	Hopf monoid	Independence
Permutahedron	Π	Classical
Associahedron	F	Monotone
Cyclohedron	C	Conditional monotone
$\vdots$	$\vdots$	$\vdots$

Joint work with Cesar Ceballos, Adrián Celestino and Franz Lehner (ANR-FWF International Cooperation Project PAGCAP - *Beyond Permutahedra and Associahedra: Geometry, Combinatorics, Algebra, and Probability*).

Merci!

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