On the solutions of universal differential equations by noncommutative Picard-Vessiot theory [1]

V.C. Bui[1], V. Hoang Ngoc Minh[2,3], Q.H. Ngô[4], V. Nguyen Dinh[3].

[1]Hue University of Sciences, 77 - Nguyen Hue street - Hue city, Vietnam.
[2]Université Lille, 1 Place Déliot, 59024 Lille, France.
[3]LIPN-UMR 7030, 99 avenue Jean-Baptiste Clément, 93430 Villetaneuse, France.
[4]University of Hai Phong, 171, Phan Dang Luu, Kien An, Hai Phong, Viet Nam.

Journées annuelles du GT CombAlg, 3 - 4 Juillet, 2023, Paris.

1. arXiv:2209.08596

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2. See also DOI : 10.5802/cml.59 (*On the solutions of the universal differential equation with three regular singularities*) and my talk at the XV International Workshop Lie Theory and Its Applications in Physics, 19–25 June 2023, Varna, Bulgaria.

INTRODUCING EXAMPLE

Knizhnik-Zamolodchikov differential equations

$(\mathcal{H}(\mathcal{V}), 1_{\mathcal{H}(\mathcal{V})})$: ring of holomorphic functions over $\mathcal{V} = \widetilde{\mathbb{C}^n_*}$, the universal covering of $\mathbb{C}^n_* := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n | z_i \neq z_j \text{ for } i \neq j\}$.
$\mathcal{H}(\mathcal{V})\langle\langle \mathcal{T}_n \rangle\rangle$: ring of series over $\mathcal{T}_n := \{t_{i,j}\}_{1 \leq i < j \leq n}$ with coefficients in $\mathcal{H}(\mathcal{V})$ and is equipped the disc. topo., *i.e.* for any $S, T \in \mathcal{H}(\mathcal{V})\langle\langle \mathcal{T}_n \rangle\rangle$,

$$d(S,T) = 2^{\varpi(S-T)}, \quad \text{where} \quad \varpi(S) = \begin{cases} +\infty & \text{if} \quad S = 0, \\ \inf\limits_{w \in \mathrm{supp}(S)} |w| & \text{if} \quad S \neq 0. \end{cases}$$

$$(KZ_n) \quad \mathbf{d}F = \Omega_n F, \quad \text{where} \quad \Omega_n(z) := \sum_{1 \leq i < j \leq n} \frac{t_{i,j}}{2\mathrm{i}\pi} d\log(z_j - z_i).$$

Example ($\mathcal{T}_2 = \{t_{1,2}\}$, KZ_2 : trivial case)

With $\Omega_2(z) = (t_{1,2}/2\mathrm{i}\pi) d\log(z_1 - z_2)$, $\mathbf{d}F = \Omega_2 F$ admits, in $\mathcal{H}(\widetilde{\mathbb{C}^2_*})\langle\langle \mathcal{T}_2 \rangle\rangle$, $F(z_1, z_2) = e^{t_{1,2}/2\mathrm{i}\pi \log(z_1 - z_2)} = (z_1 - z_2)^{t_{1,2}/2\mathrm{i}\pi}$ as solution.

For $n > 2$, solutions of (KZ_n) can be computed by iterations of pointwise convergence, for the disc. topo. over $\mathcal{H}(\mathcal{V})\langle\langle \mathcal{T}_n \rangle\rangle$.

Example (Picard's iteration)

$$F_0(z_0, z) = 1_{\mathcal{H}(\mathcal{V})} \quad \text{and} \quad F_l(z_0, z) = F_{l-1}(z_0, z) + \int_{z_0}^{z} \Omega_n(s) F_{l-1}(z_0, s).$$

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Integrability and dévissage

According to Drinfel'd, (KZ_n) is completely integrable if Ω_n is flat, *i.e.*

$$\mathbf{d}\Omega_n - \Omega_n \wedge \Omega_n = 0.$$

It turns out that this condition induces the following quadratic relations among $\{t_{i,j}\}_{1\leq i<j\leq n}$ (Kohno's lemma) :

$$\mathcal{R}_n = \begin{cases} [t_{i,k} + t_{j,k}, t_{i,j}] = 0 & \text{for distinct } i,j,k & \text{and } 1 \leq i < j < k \leq n, \\ [t_{i,j} + t_{i,k}, t_{j,k}] = 0 & \text{for distinct } i,j,k & \text{and } 1 \leq i < j < k \leq n, \\ [t_{i,j}, t_{k,l}] = 0 & \text{for distinct } i,j,k,l & \text{and } \begin{cases} 1 \leq i < j \leq n, \\ 1 \leq k < l \leq n, \end{cases} \end{cases}$$

generating the Lie ideal of relators, $\mathcal{J}_{\mathcal{R}_n}$. Solutions of KZ_n can be then iteratively computed over $\mathcal{H}(\mathcal{V})\langle\langle\mathcal{T}_n\rangle\rangle$ and then over $\mathcal{H}(\mathcal{V})\langle\langle\mathcal{T}_n\rangle\rangle/\mathcal{J}_{\mathcal{R}_n}$.

For $z_n \to z_{n-1}$

$$\Omega_n(z) = \underbrace{\sum_{1\leq i<j\leq n-1} \frac{t_{i,j}}{2\mathrm{i}\pi}\frac{d(z_j - z_i)}{z_j - z_i}}_{\Omega_{n-1}(z)\longleftrightarrow\mathcal{T}_{n-1}} + \underbrace{\sum_{j=1}^{n-2} \frac{t_{i,n}}{2\mathrm{i}\pi}\frac{d(z_n - z_j)}{z_n - z_j} + \frac{t_{n-1,n}}{2\mathrm{i}\pi}\frac{d(z_n - z_{n-1})}{z_n - z_{n-1}}}_{\text{c.f. Lappo-Danilevsky's hyperlogarithms}}.$$

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KZ_3 : simplest non-trivial case, $\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$

$$\Omega_3 = (2\mathrm{i}\pi)^{-1}[t_{1,2}d\log(z_1 - z_2) + t_{1,3}d\log(z_1 - z_3) + t_{2,3}d\log(z_2 - z_3)].$$

$\mathbf{d}F = \Omega_3 F$ can be computed by the sequence $\{V_l\}_{l\geq 0}$ (on $\mathcal{H}(\widetilde{\mathbb{C}^3_*})\langle\langle\mathcal{T}_3\rangle\rangle$) defined by $V_0(z) = e^{t_{1,2}/2\mathrm{i}\pi\log(z_1-z_2)}$ and recursively

$$V_l(z) = V_0(z)\int_0^z e^{-t_{1,2}/2\mathrm{i}\pi\log(s_1-s_2)}\tilde{\Omega}_2(s)V_{l-1}(s),$$

where $\tilde{\Omega}_2(z) = (2\mathrm{i}\pi)^{-1}[t_{1,3}d\log(z_1 - z_3) + t_{2,3}d\log(z_2 - z_3)]$.

$$\begin{aligned}
\sum_{l\geq 0} V_l &= V_0 G, \text{ where} \\
G(z) &= \sum_{l\geq 0}\sum_{i_1,\ldots,i_l\in\{0,2\}}\int_0^z \omega_{i_1,3}(s_1)\varphi^{(0,s_1)}(t_{i_1,3})\ldots\int_0^{s_{l-1}}\omega_{i_l,3}(s_l)\varphi^{(0,s_l)}(t_{i_l,3}) \\
&= \sum_{l\geq 0}\sum_{i_1,\ldots,i_l\in\{0,2\}}\int_0^z \omega_{i_1,3}(s_1)\ldots\int_0^{s_{l-1}}\omega_{i_l,3}(s_l)\underbrace{\varphi^{(0,s_1)}(t_{i_1,3})\ldots\varphi^{(0,s_l)}(t_{i_l,3})}_{=\varphi^{(0,z)}(t_{i_1,3}\ldots t_{i_l,3})}
\end{aligned}$$

and φ is the chronological conc-morphism of $\mathbb{C}\langle\mathcal{T}_3\rangle$ defined, for a subdivision $(0, s_1, \ldots, s_k, z)$ of $0 \rightsquigarrow z$, by $\varphi^{(0,z)}(t_{i,3}) = e^{\mathrm{ad}_{-t_{1,2}/2\mathrm{i}\pi\log(z_1-z_2)}}t_{i,3}$ ($i = 1$ or 2) and $\varphi^{(0,z)}(t_{i_1,3}\ldots t_{i_l,3}) = \varphi^{(0,s_1)}(t_{i_1,3})\ldots\varphi^{(0,s_l)}(t_{i_l,3})$.

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&= \sum_{l\geq 0}\sum_{i_1,\ldots,i_l\in\{0,2\}}\int_0^z \omega_{i_1,3}(s_1)\ldots\int_0^{s_{l-1}}\omega_{i_l,3}(s_l)\underbrace{\varphi^{(0,s_1)}(t_{i_1,3})\ldots\varphi^{(0,s_l)}(t_{i_l,3})}_{=\varphi^{(0,z)}(t_{i_1,3}\ldots t_{i_l,3})}
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Noncommutative generating series of polylogarithms

$$\mathbf{d}G = (\varphi(t_{1,3})\omega_{1,3} + \varphi(t_{2,3})\omega_{2,3})G, \quad \begin{cases} \omega_{1,3}(z) = (2\mathrm{i}\pi)^{-1} d\log(z_1 - z_3), \\ \omega_{2,3}(z) = (2\mathrm{i}\pi)^{-1} d\log(z_2 - z_3). \end{cases}$$

In $(P_{1,2}) : z_1 - z_2 = 1, \varphi \equiv \mathrm{Id}$ and then, putting $(z_1, z_2, z_3) = (1, 0, s)$,

$$\mathbf{d}G = (x_1\omega_1 + x_0\omega_0)G, \quad \begin{cases} x_0 = t_{1,3}/2\mathrm{i}\pi, & \omega_0(s) = d\log(s), \\ x_1 = -t_{2,3}/2\mathrm{i}\pi, & \omega_1(s) = -d\log(1-s). \end{cases}$$

This can be solved by Picard's iteration :

$$C_{s_0 \rightsquigarrow s} := \sum_{w \in X^*} \alpha_{s_0}^s(w) w, \text{ where } \alpha_{s_0}^s(w) = \begin{cases} 1_{\mathcal{H}(B)} & \text{if} \quad w = 1_{X^*}, \\ \int_{s_0}^{s} \omega_i(t) \alpha_{s_0}^t(u) & \text{if} \quad w = x_i u, \end{cases}$$

where $(X^*, 1_{X^*})$ is the free monoid generated by $X := \{x_0, x_1\}$ and $(\mathcal{H}(B), 1_{\mathcal{H}(B)})$ is the ring of holomorphic functions over $B := \widetilde{\mathbb{C} \setminus \{0, 1\}}$.
Let $\{P_l\}_{l \in \mathcal{L}yn X}$ be the basis of $\mathcal{L}ie_{\mathcal{H}(B)}\langle X \rangle$, in duality with the pure transcendence basis $\{S_l\}_{l \in \mathcal{L}yn X}$ of $(\mathcal{H}(B)\langle X \rangle, ш)$.

Let $\mathrm{Li}_\bullet$ be the ш-character defined, over the algebraic basis $\mathcal{L}yn X$, by

$$\mathrm{Li}_{1_{X^*}} = 1_{\mathcal{H}(B)}, \quad \mathrm{Li}_{x_0}(s) = \log(s), \quad \mathrm{Li}_{x_1}(s) = -\log(1-s) \quad \text{and}$$

$$\mathrm{Li}_{x_i w}(s) = \int_0^s \omega_i(\sigma) \mathrm{Li}_w(\sigma), \quad \text{for} \quad x_i w \in \mathcal{L}yn X \setminus X \subset x_0 X^* x_1.$$

$\{\mathrm{Li}_l\}_{l \in \mathcal{L}yn X}$ (resp. $\{\mathrm{Li}_w\}_{w \in X^*}$) are $\mathcal{H}(B)$-algebraically (resp. linearly) free.

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Solution of KZ_3 using n.g.s. of polylogarithms

$$\mathrm{L} := \sum_{w\in X^*} \mathrm{Li}_w w = \prod_{l\in\mathcal{L}yn X}^{\searrow} e^{\mathrm{Li}_{s_l} P_l} \quad \text{and} \quad \Phi_{KZ} := \prod_{l\in\mathcal{L}yn X\setminus X}^{\searrow} e^{\mathrm{Li}_{s_l}(1) P_l}.$$

L satisfies also $\mathbf{d}G = (x_1\omega_1 + x_0\omega_0)G$ and then $\mathrm{L}(s) = C_{s_0\rightsquigarrow s}\mathrm{L}^{-1}(s_0)$.
For $s_0 \to 0$, $\mathrm{L}(s)$ normalizes $C_{s_0\rightsquigarrow s}$ and $\mathrm{L}(s_0)$ is a counter term.

$$\lim_{s\to 0} \mathrm{L}(s)e^{-x_0\log s} = 1 \quad \text{and} \quad \lim_{z\to 1} e^{x_1\log(1-s)}\mathrm{L}(s) = \Phi_{KZ},$$
$$\iff \mathrm{L}(s) \sim_0 e^{x_0\log s} \quad \text{and} \quad \mathrm{L}(s) \sim_1 e^{-x_1\log(1-s)}\Phi_{KZ}.$$

Let $g : s \longmapsto \dfrac{s-z_2}{z_1-z_2}$ be the homo. transf. mapping $\{z_2, z_1\}$ to $\{0,1\}$.

Then $\mathrm{L}(g(s)) = \mathrm{L}\left(\dfrac{s-z_2}{z_1-z_2}\right)$ is a particular solution of (KZ_3) in $(P_{1,2})$.

So does[3] $\mathrm{L}\left(\dfrac{s-z_2}{z_1-z_2}\right)(z_1-z_2)^{(t_{1,2}+t_{1,3}+t_{2,3})/2\mathrm{i}\pi}$.

Since $[t_{1,2}+t_{2,3}+t_{1,3}, t] = 0$, for $t \in \mathcal{T}_3$, then $(z_1-z_2)^{(t_{1,2}+t_{2,3}+t_{1,3})/2\mathrm{i}\pi}$ commutes with t and then with $\mathcal{H}(\widetilde{\mathbb{C}^3_*})\langle\langle\mathcal{T}_3\rangle\rangle$. Thus, (KZ_3) also admits $(z_1-z_2)^{(t_{1,2}+t_{1,3}+t_{2,3})/2\mathrm{i}\pi}\mathrm{L}\left(\dfrac{s-z_2}{z_1-z_2}\right)$ as a particular solution in $(P_{1,2})$.

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3. $(z_1-z_2)^{(t_{1,2}+t_{2,3}+t_{1,3})/2\mathrm{i}\pi} = e^{((t_{1,2}+t_{2,3}+t_{1,3})/2\mathrm{i}\pi)\log(z_1-z_2)}$, being independent on $z_3 = s$ and then belonging to the differential Galois group of KZ_3.

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$$\mathrm{L} := \sum_{w\in X^*} \mathrm{Li}_w w = \prod_{l\in\mathcal{L}yn X}^{\searrow} e^{\mathrm{Li}_{s_l} P_l} \quad \text{and} \quad \Phi_{KZ} := \prod_{l\in\mathcal{L}yn X\setminus X}^{\searrow} e^{\mathrm{Li}_{s_l}(1) P_l}.$$

L satisfies also $\mathbf{d}G = (x_1\omega_1 + x_0\omega_0)G$ and then $\mathrm{L}(s) = C_{s_0\rightsquigarrow s}\mathrm{L}^{-1}(s_0)$.
For $s_0 \to 0$, $\mathrm{L}(s)$ normalizes $C_{s_0\rightsquigarrow s}$ and $\mathrm{L}(s_0)$ is a counter term.

$$\lim_{s\to 0} \mathrm{L}(s)e^{-x_0\log s} = 1 \quad \text{and} \quad \lim_{z\to 1} e^{x_1\log(1-s)}\mathrm{L}(s) = \Phi_{KZ},$$
$$\iff \mathrm{L}(s) \sim_0 e^{x_0\log s} \quad \text{and} \quad \mathrm{L}(s) \sim_1 e^{-x_1\log(1-s)}\Phi_{KZ}.$$

Let $g : s \longmapsto \dfrac{s-z_2}{z_1-z_2}$ be the homo. transf. mapping $\{z_2, z_1\}$ to $\{0,1\}$.

Then $\mathrm{L}(g(s)) = \mathrm{L}\left(\dfrac{s-z_2}{z_1-z_2}\right)$ is a particular solution of (KZ_3) in $(P_{1,2})$.

So does[3] $\mathrm{L}\left(\dfrac{s-z_2}{z_1-z_2}\right)(z_1-z_2)^{(t_{1,2}+t_{1,3}+t_{2,3})/2\mathrm{i}\pi}$.

Since $[t_{1,2}+t_{2,3}+t_{1,3}, t] = 0$, for $t \in \mathcal{T}_3$, then $(z_1-z_2)^{(t_{1,2}+t_{2,3}+t_{1,3})/2\mathrm{i}\pi}$ commutes with t and then with $\mathcal{H}(\widetilde{\mathbb{C}^3_*})\langle\langle\mathcal{T}_3\rangle\rangle$. Thus, (KZ_3) also admits $(z_1-z_2)^{(t_{1,2}+t_{1,3}+t_{2,3})/2\mathrm{i}\pi}\mathrm{L}\left(\dfrac{s-z_2}{z_1-z_2}\right)$ as a particular solution in $(P_{1,2})$.

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3. $(z_1-z_2)^{(t_{1,2}+t_{2,3}+t_{1,3})/2\mathrm{i}\pi} = e^{((t_{1,2}+t_{2,3}+t_{1,3})/2\mathrm{i}\pi)\log(z_1-z_2)}$, being independent on $z_3 = s$ and then belonging to the differential Galois group of KZ_3.

Solution of KZ_3 using n.g.s. of polylogarithms

$$\mathrm{L} := \sum_{w\in X^*} \mathrm{Li}_w w = \prod_{l\in\mathcal{L}yn X}^{\searrow} e^{\mathrm{Li}_{s_l} P_l} \quad \text{and} \quad \Phi_{KZ} := \prod_{l\in\mathcal{L}yn X\setminus X}^{\searrow} e^{\mathrm{Li}_{s_l}(1) P_l}.$$

L satisfies also $\mathbf{d}G = (x_1\omega_1 + x_0\omega_0)G$ and then $\mathrm{L}(s) = C_{s_0\rightsquigarrow s}\mathrm{L}^{-1}(s_0)$.
For $s_0 \to 0$, $\mathrm{L}(s)$ normalizes $C_{s_0\rightsquigarrow s}$ and $\mathrm{L}(s_0)$ is a counter term.

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So does[3] $\mathrm{L}\left(\dfrac{s-z_2}{z_1-z_2}\right)(z_1-z_2)^{(t_{1,2}+t_{1,3}+t_{2,3})/2\mathrm{i}\pi}$.

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ALGEBRAIC COMBINATORIAL FRAMWORKS

Some notations on free Lie algebra and on shuffle algebra

$\mathcal{T}_n$ generates the free monoid $(\mathcal{T}_n^*, 1_{\mathcal{T}_n^*})$ in which $\mathtt{conc}$ (resp. $\Delta_{\mathtt{conc}}$) denotes the concatenation product (resp. co-product).
$\mathcal{A}\langle\langle\mathcal{T}_n\rangle\rangle$ and $\mathcal{A}\langle\mathcal{T}_n\rangle$ (resp. $\mathcal{L}ie_{\mathcal{A}}\langle\langle\mathcal{T}_n\rangle\rangle$ and $\mathcal{L}ie_{\mathcal{A}}\langle\mathcal{T}_n\rangle$) are the (resp. Lie) algebras of series and of polynomials over $\mathcal{T}_n$ with coefficients in the integral ring $\mathcal{A}$.

Let r be the right normed bracketing defined by $r(1_{\mathcal{T}_n^*}) = 0$ and $r(t_1 \dots t_k) = [t_1, [\dots, [t_{k-1}, t_k] \dots] = \mathrm{ad}_{t_1} \dots \mathrm{ad}_{t_{k-1}} t_k$.
Let the product $\sqcup\!\sqcup$ be defined, for any $x, y \in \mathcal{T}_n$ and $u, v \in \mathcal{T}_n^*$, by $u \sqcup\!\sqcup 1_{\mathcal{T}_n^*} = 1_{\mathcal{T}_n^*} \sqcup\!\sqcup u = u$ and $xu \sqcup\!\sqcup yv = x(u \sqcup\!\sqcup yv) + y(v \sqcup\!\sqcup xu)$, or equivalently, $\Delta_{\sqcup\!\sqcup} x = x \otimes 1_{\mathcal{T}_n^*} + 1_{\mathcal{T}_n^*} \otimes x$.

Let $|v|_t$ be the number of occurrences of t in $v = t_1 \dots t_m \in \mathcal{T}_n^*$ and

$$\tilde{v} = t_m \dots t_1, \quad a(v) = (-1)^{|v|} \tilde{v}, \quad \hat{v} = \mathop{\sqcup\!\sqcup}_{t \in \mathcal{T}_n} t^{|v|_t}.$$

Hence, for any S and $R \in \mathcal{A}\langle\langle\mathcal{T}_n\rangle\rangle$,

$$a(S) = \sum_{w \in \mathcal{T}_n^*} \langle S|w\rangle a(w) \quad \text{and} \quad \begin{cases} a(SR) &= a(R)a(S), \\ a(S \sqcup\!\sqcup R) &= a(S) \sqcup\!\sqcup a(R). \end{cases}$$

If $\langle S|1_{\mathcal{T}_n^*}\rangle = 1$ then $a(S)$ is its inverse for $\mathtt{conc}$, *i.e.*
$Sa(S) = a(S)S = 1_{\mathcal{T}_n^*}$ and then $a(e^L) = e^{-L}$, for $L \in \mathcal{L}ie_{\mathcal{A}}\langle\langle\mathcal{T}_n\rangle\rangle$.

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Let the product $ш$ be defined, for any $x, y \in \mathcal{T}_n$ and $u, v \in \mathcal{T}_n^*$, by $u ш 1_{\mathcal{T}_n^*} = 1_{\mathcal{T}_n^*} ш u = u$ and $xu ш yv = x(u ш yv) + y(v ш xu)$, or equivalently, $\Delta_{ш} x = x \otimes 1_{\mathcal{T}_n^*} + 1_{\mathcal{T}_n^*} \otimes x$.

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PBW-CQMM theorems and bases in $(\mathcal{A}\langle\mathcal{T}_n\rangle, ⧢, 1_{\mathcal{T}_n^*}, \Delta_{\mathrm{conc}})$

Let $\{b_i\}_{i\in I}$ be a basis of $\mathcal{L}ie_{\mathcal{A}}\langle\mathcal{T}_n\rangle$. Let [4] $\{b^\alpha\}_{\alpha\in\mathbb{N}^{(I)}}$ be the associated PBW basis of the enveloping algebra $\mathcal{U}(\mathcal{L}ie_{\mathcal{A}}\langle\mathcal{T}_n\rangle)$.

By CQMM theorem, the dual basis $\{\check{b}^\alpha\}_{\alpha\in\mathbb{N}^{(I)}}$ of the associative commutative algebra $\mathcal{U}(\mathcal{L}ie_{\mathcal{A}}\langle\mathcal{T}_n\rangle)^\vee$ is constructed as follows [5]

$$\check{b}^\alpha\check{b}^\beta = \frac{(\alpha+\beta)!}{\alpha!\beta!}\check{b}^{\alpha+\beta} \quad \text{and} \quad \check{b}_{\alpha(i_1)m_{i_1}+\cdots+\alpha(i_k)m_{i_k}} = \frac{\check{b}^{\alpha(i_1)}_{m_{i_1}}}{\alpha(i_1)!}\cdots\frac{\check{b}^{\alpha(i_k)}_{m_{i_k}}}{\alpha(i_k)!}.$$

$$\langle b^\beta|\check{b}^\alpha\rangle = \delta_{\alpha,\beta} \quad \text{and, on } \mathrm{End}(\mathcal{U}(\mathcal{L}ie_{\mathcal{A}}\langle\mathcal{T}_n\rangle)), \quad \mathrm{Id}_{\mathcal{U}(\mathcal{L}ie_{\mathcal{A}}\langle\mathcal{T}_n\rangle)} = \prod_{i\in I} e^{\check{b}^{e_i}\otimes b^{e_i}}.$$

If $\mathcal{U}(\mathcal{L}ie_{\mathcal{A}}\langle\mathcal{T}_n\rangle)$ is endowed with the PBW basis $\{P_w\}_{w\in\mathcal{T}_n^*}$, containing the basis $\{P_l\}_{l\in\mathcal{L}yn\mathcal{T}_n}$ of $\mathcal{L}ie_{\mathcal{A}}\langle\mathcal{T}_n\rangle$, and $\mathcal{U}(\mathcal{L}ie_{\mathcal{A}}\langle\mathcal{T}_n\rangle)^\vee$ with $\{S_w\}_{w\in\mathcal{T}_n^*}$, containing the transcendence basis $\{S_l\}_{l\in\mathcal{L}yn\mathcal{T}_n}$ of $(\mathcal{A}\langle\mathcal{T}_n\rangle, ⧢, 1_{\mathcal{T}_n^*})$ then

$$\mathcal{D}_{\mathcal{T}_n} := \sum_{w\in\mathcal{T}_n^*} w\otimes w = \sum_{w\in\mathcal{T}_n^*} S_w\otimes P_w = \overset{\searrow}{\prod_{\mathcal{L}yn\mathcal{T}_n}} e^{S_l\otimes P_l}.$$

---

4. using the elementary multiindex, $m_i \in \mathbb{N}^{(I)}$, defined by $m_i(j) = \delta_{i,j}$ (for $i,j\in I$) and, the multiindex notation, *i.e.*

$$\forall\alpha\in\mathbb{N}^{(I)}, \quad \mathrm{supp}(\alpha)\subset\{i_1,\cdots,i_n\}, \quad b^\alpha = b_{i_1}^{\alpha(i_1)}\cdots b_{i_n}^{\alpha(i_n)}.$$

5. $\forall\alpha\in\mathbb{N}^{(I)}, \alpha! = \prod_{i\in I}\alpha_i!$.

PBW-CQMM theorems and bases in $(\mathcal{A}\langle\mathcal{T}_n\rangle, ⧢, 1_{\mathcal{T}_n^*}, \Delta_{\mathrm{conc}})$

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From partitioning of $\mathcal{T}_n$ to Lazard's elimination ...

Let $T_n := \{t_{j,n}\}_{1\leq j\leq n-1}$ s.t. $\mathcal{T}_n = T_n \sqcup \mathcal{T}_{n-1}$.

Example

$\mathcal{T}_4 = \{t_{1,2}, t_{1,3}, t_{1,4}, t_{2,3}, t_{2,4}, t_{3,4}\}$, one has $T_4 = \{t_{1,4}, t_{2,4}, t_{3,4}\}$ and $\mathcal{T}_3$.
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Let $\mathcal{I}_n$ be the Lie subalgebra generated by $\{r(vt)\}_{\substack{v\in T_n^* \\ t\in\mathcal{T}_{n-1}}}$. Then

$$\mathcal{L}ie_{\mathcal{A}}\langle\mathcal{T}_n\rangle = \mathcal{L}ie_{\mathcal{A}}\langle T_n\rangle \oplus \mathcal{I}_n.$$

As Lie algebra, $\mathcal{I}_n$ is obviously a Leibniz algebra[6] and then $\mathcal{I}_n^\vee$ is generated by $\{a(vt)\}_{\substack{v\in T_n^* \\ t\in\mathcal{T}_{n-1}}}$, being a Zinbiel subalgebra of $(\mathcal{A}\langle\mathcal{T}_n\rangle, \frac{ш}{2})$, where the half-shuffle product is defined by $(t \in \mathcal{T}_n, R, H \in \mathcal{A}\langle\langle\mathcal{T}_n\rangle\rangle)$

$$1_{\mathcal{T}_n^*} \frac{ш}{2} (tH) = 0 \quad \text{and} \quad (tH)\frac{ш}{2} R = \begin{cases} tH & \text{if} \quad R = 1_{\mathcal{T}_n^*}, \\ t(H ш R) & \text{if} \quad R \neq 1_{\mathcal{T}_n^*}. \end{cases}$$

Example $(\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\})$

$$\begin{aligned} (t_{1,3}t_{1,2})\frac{ш}{2} t_{2,3} &= t_{1,3}(t_{1,2} ш t_{2,3}) &= t_{1,3}t_{1,2}t_{2,3} + t_{1,3}t_{2,3}t_{1,2}, \\ (t_{1,3}t_{1,2}^*)\frac{ш}{2} t_{2,3} &= t_{1,3}(t_{1,2}^* ш t_{2,3}) &= t_{1,3}t_{1,2}^*t_{2,3}t_{1,2}^*.^{7} \end{aligned}$$

6. *i.e.* $\forall x, y, z \in \mathcal{I}_n$, the identity $[x,[y,z]] = [[x,y],z] - [[x,z],y]$ holds.
7. $\forall S \in \mathcal{A}\langle\langle\mathcal{T}_n\rangle\rangle, \quad \langle S|1_{\mathcal{T}_n^*}\rangle = 0, \quad S^* = 1_{\mathcal{T}_n^*} + S + S^2 + \dots$.

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where the half-shuffle product is defined by ($t \in \mathcal{T}_n, R, H \in \mathcal{A}\langle\langle\mathcal{T}_n\rangle\rangle$)

$$1_{\mathcal{T}_n^*} \underline{\sqcup\!\sqcup}_{\frac{1}{2}} (tH) = 0 \quad \text{and} \quad (tH) \underline{\sqcup\!\sqcup}_{\frac{1}{2}} R = \begin{cases} tH & \text{if} \quad R = 1_{\mathcal{T}_n^*}, \\ t(H \sqcup\!\sqcup R) & \text{if} \quad R \neq 1_{\mathcal{T}_n^*}. \end{cases}$$

Example ($\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$)

$$\begin{aligned} (t_{1,3}t_{1,2}) \underline{\sqcup\!\sqcup}_{\frac{1}{2}} t_{2,3} &= t_{1,3}(t_{1,2} \sqcup\!\sqcup t_{2,3}) &&= t_{1,3}t_{1,2}t_{2,3} + t_{1,3}t_{2,3}t_{1,2}, \\ (t_{1,3}t_{1,2}^*) \underline{\sqcup\!\sqcup}_{\frac{1}{2}} t_{2,3} &= t_{1,3}(t_{1,2}^* \sqcup\!\sqcup t_{2,3}) &&= t_{1,3}t_{1,2}^*t_{2,3}t_{1,2}^*.^{7} \end{aligned}$$

---

6. *i.e.* $\forall x, y, z \in \mathcal{I}_n$, the identity $[x,[y,z]] = [[x,y],z] - [[x,z],y]$ holds.
7. $\forall S \in \mathcal{A}\langle\langle\mathcal{T}_n\rangle\rangle, \quad \langle S|1_{\mathcal{T}_n^*}\rangle = 0, \quad S^* = 1_{\mathcal{T}_n^*} + S + S^2 + \dots$.

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More generally, for any $u_i \in \mathcal{T}_n^+$ and $l_i \in \mathcal{L}yn\mathcal{T}_n, 1 \leq i \leq k \in \mathbb{N}_{\geq 1}$,

$$\underset{i=1}{\overset{k}{\sqcup\!\sqcup}} u_i = \sum_{\sigma \in \mathfrak{S}_k} u_{\sigma(1)} \underset{\frac{}{2}}{\sqcup\!\sqcup} (\ldots (\underset{\frac{}{2}}{\sqcup\!\sqcup} u_{\sigma(k)})), \quad \underset{i=1}{\overset{k}{\sqcup\!\sqcup}} S_{l_i} = \sum_{\sigma \in \mathfrak{S}_k} S_{l_{\sigma(1)}} \underset{\frac{}{2}}{\sqcup\!\sqcup} (\ldots (\underset{\frac{}{2}}{\sqcup\!\sqcup} S_{l_{\sigma(k)}})).$$

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Other dual bases in bialgebras

$$\begin{aligned}
\mathcal{B} &:= \{\mathrm{ad}^{k_1}_{-T_n} t_1 \dots \mathrm{ad}^{k_p}_{-T_n} t_p\}^{k_1,\dots,k_p\geq 0, p\geq 1}_{t_1,\dots,t_p\in\mathcal{T}_{n-1}} \\
&= \{(-1)^{|v_1\dots v_k|} r(v_1 t_1)\cdots r(v_p t_p)\}^{p\geq 1}_{\substack{v_1,\dots,v_p\in T_n^* \\ t_1,\dots,t_p\in\mathcal{T}_{n-1}}} , \\
\mathcal{B}^\vee &:= \{(-t_1 T_n^{k_1}) ⧢ \cdots ⧢ (-t_p T_n^{k_p})\}^{k_1,\dots,k_p\geq 0, p\geq 1}_{t_1,\dots,t_p\in\mathcal{T}_{n-1}} \\
&= \{a(u_1 t_1) ⧢ \cdots ⧢ a(u_p t_p)\}^{p\geq 1}_{\substack{u_1,\dots,u_p\in T_n^* \\ t_1,\dots,t_p\in\mathcal{T}_{n-1}}} .
\end{aligned}$$

For any $v_1, v_2 \in T_n^*$ and $t_1, t_2 \in \mathcal{T}_{n-1}$, $\langle a(v_1 t_1) | r(v_2 t_2)\rangle = \delta^{v_2 t_2}_{v_1 t_1}$. Hence, $\mathcal{I}_n \simeq (\mathrm{span}_{\mathcal{A}}\{r(vt)\}_{\substack{v\in T_n^* \\ t\in\mathcal{T}_{n-1}}}, [,])$ and then, by duality,

$\mathcal{I}_n^\vee \simeq (\mathrm{span}_{\mathcal{A}}\{-tu\}_{\substack{u\in T_n^* \\ t\in\mathcal{T}_{n-1}}}, ⧢) \simeq (\mathrm{span}_{\mathcal{A}}\{a(vt)\}_{\substack{v\in T_n^* \\ t\in\mathcal{T}_{n-1}}}, \frac{⧢}{2})$.

For any $u_1, \dots, u_p, v_1, \dots, v_p \in T_n^*$ and $x_1, \dots, x_p, y_1, \dots, y_p \in \mathcal{T}_{n-1}$, $\langle a(u_1 x_1) \frac{⧢}{2} (\cdots \frac{⧢}{2} a(u_p x_p) \dots)) | r(v_1 y_1) \dots r(v_p y_p)\rangle = \delta^{v_1 y_1 \dots v_p y_p}_{u_1 x_1 \dots u_p x_p}$. Hence,

$$\begin{aligned}
\mathcal{U}(\mathcal{I}_n) &\simeq \mathrm{span}_{\mathcal{A}}\{(-1)^{|v_1\dots v_k|} r(v_1 t_1)\cdots r(v_p t_p)\}^{p\geq 1}_{\substack{v_1,\dots,v_p\in T_n^* \\ t_1,\dots,t_p\in\mathcal{T}_{n-1}}} , \\
\mathcal{U}(\mathcal{I}_n)^\vee &\simeq \mathrm{span}_{\mathcal{A}}\{a(u_1 t_1) ⧢ \cdots ⧢ a(u_p t_p)\}^{p\geq 1}_{\substack{u_1,\dots,u_p\in T_n^* \\ t_1,\dots,t_p\in\mathcal{T}_{n-1}}} \\
&\simeq \mathrm{span}_{\mathcal{A}}\{a(v_1 t_1) \frac{⧢}{2} (\cdots \frac{⧢}{2} a(v_p t_p) \dots))\}^{p\geq 1}_{\substack{v_1,\dots,v_p\in T_n^* \\ t_1,\dots,t_p\in\mathcal{T}_{n-1}}} .
\end{aligned}$$

Proposition

$\mathcal{U}(\mathcal{L}ie_{\mathcal{A}}\langle\mathcal{T}_n\rangle)$ and $\mathcal{U}(\mathcal{L}ie_{\mathcal{A}}\langle\mathcal{T}_n\rangle)^\vee$ admit $T_n^*\mathcal{B}$ and $T_n^* ⧢ \mathcal{B}^\vee$ as dual bases.

Factorizations of grouplike series in bialgebras

If $\mathcal{A}\langle\langle \mathcal{T}_n \rangle\rangle \ni S$ is grouplike then, as image of $\mathcal{D}_{\mathcal{T}_n}$ by $S \otimes \mathrm{Id}$,

$$S = \sum_{w \in \mathcal{T}_n^*} \langle S|w\rangle w = \sum_{w \in \mathcal{T}_n^*} \langle S|S_w\rangle P_w = \overset{\searrow}{\prod_{\mathcal{L}yn\mathcal{T}_n}} e^{\langle S|S_l\rangle \otimes P_l}.$$

$$S^{-1} = a(S) = \overset{\swarrow}{\prod_{l \in \mathcal{L}yn\mathcal{T}_n}} a(e^{\langle S|S_l\rangle P_l}) = \overset{\swarrow}{\prod_{l \in \mathcal{L}yn\mathcal{T}_n}} e^{-\langle S|S_l\rangle P_l}.$$

Theorem

The diagonal series can be factorized as follows[8]

$$\mathcal{D}_{\mathcal{T}_n} = \mathcal{D}_{T_n}\Big(1_{\mathcal{T}_n^*} \otimes 1_{\mathcal{T}_n^*} + \sum_{k \geq 1} \sum_{\substack{v_1,\ldots,v_k \in T_n^* \\ t_1,\ldots,t_k \in \mathcal{T}_{n-1}}} a(v_1 t_1) \underset{2}{\sqcup\!\sqcup} (\cdots \underset{2}{\sqcup\!\sqcup} a(v_k t_k) \ldots)) \otimes r(v_1 t_1) \ldots r(v_k t_k)\Big).$$

Ordering by $t_{1,k} \succ \ldots \succ t_{k-1,k}$, $T_2 \succ \ldots \succ T_n$, $\mathcal{L}yn T_2 \succ \ldots \succ \mathcal{L}yn T_n$, *it can be also factorized as follows*

$$\mathcal{D}_{\mathcal{T}_n} = \overset{\searrow}{\prod_{l \in \mathcal{L}yn\mathcal{T}_n}} e^{S_l \otimes P_l} = \mathcal{D}_{\mathcal{T}_{n-1}}\Big(\overset{\searrow}{\prod_{\substack{l = l_1 l_2 \\ l_2 \in \mathcal{L}yn\mathcal{T}_{n-1}, l_1 \in \mathcal{L}yn T_n}}} e^{S_l \otimes P_l}\Big)\mathcal{D}_{T_n}.$$

8. By the standard factorization, $\mathcal{L}yn\mathcal{T}_{n-1} \succ \mathcal{L}yn T_n . \mathcal{L}yn\mathcal{T}_{n-1} \succ \mathcal{L}yn T_n$.

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Factorizations of grouplike series in bialgebras

If $\mathcal{A}\langle\langle\mathcal{T}_n\rangle\rangle \ni S$ is grouplike then, as image of $\mathcal{D}_{\mathcal{T}_n}$ by $S \otimes \mathrm{Id}$,

$$S = \sum_{w\in\mathcal{T}_n^*} \langle S|w\rangle w = \sum_{w\in\mathcal{T}_n^*} \langle S|S_w\rangle P_w = \overset{\searrow}{\prod_{\mathcal{L}yn\mathcal{T}_n}} e^{\langle S|S_l\rangle\otimes P_l}.$$

$$S^{-1} = a(S) = \overset{\swarrow}{\prod_{l\in\mathcal{L}yn\mathcal{T}_n}} a(e^{\langle S|S_l\rangle P_l}) = \overset{\swarrow}{\prod_{l\in\mathcal{L}yn\mathcal{T}_n}} e^{-\langle S|S_l\rangle P_l}.$$

Theorem

The diagonal series can be factorized as follows[8]

$$\mathcal{D}_{\mathcal{T}_n} = \mathcal{D}_{T_n}\Big(1_{\mathcal{T}_n^*}\otimes 1_{\mathcal{T}_n^*} + \sum_{k\geq 1}\sum_{\substack{v_1,\ldots,v_k\in T_n^*\\ t_1,\ldots,t_k\in\mathcal{T}_{n-1}}} a(v_1t_1)\underset{2}{\sqcup\!\sqcup}(\cdots\underset{2}{\sqcup\!\sqcup}a(v_kt_k)\ldots))\otimes r(v_1t_1)\ldots r(v_kt_k)\Big).$$

Ordering by $t_{1,k}\succ\ldots\succ t_{k-1,k}$, $T_2\succ\ldots\succ T_n$, $\mathcal{L}yn\,T_2\succ\ldots\succ\mathcal{L}yn\,T_n$, *it can be also factorized as follows*

$$\mathcal{D}_{\mathcal{T}_n} = \overset{\searrow}{\prod_{l\in\mathcal{L}yn\mathcal{T}_n}} e^{S_l\otimes P_l} = \mathcal{D}_{\mathcal{T}_{n-1}}\Big(\overset{\searrow}{\prod_{\substack{l=l_1l_2\\ l_2\in\mathcal{L}yn\mathcal{T}_{n-1},l_1\in\mathcal{L}yn T_n}}} e^{S_l\otimes P_l}\Big)\mathcal{D}_{T_n}.$$

8. By the standard factorization, $\mathcal{L}yn\mathcal{T}_{n-1}\succ\mathcal{L}yn\,T_n.\mathcal{L}yn\mathcal{T}_{n-1}\succ\mathcal{L}yn\,T_n$.

UNIVERSAL DIFFERENTIAL EQUATION AND UNIVERSAL CONNECTION

Universal differential equation

$\mathcal{V}$: the simply connected manifold on $\mathbb{C}^n$. The pushforward (resp. pullback) of any diffeomorphism g on $\mathcal{V}$ is denoted by g_* (resp. g^*).
$\mathcal{A} = (\mathcal{H}(\mathcal{V}), \partial_1, \ldots, \partial_n)$: the differential ring ($\mathrm{Const}(\mathcal{A}) = \mathbb{C}.1_{\mathcal{H}(\mathcal{V})}$) of holomorphic functions (with $1_{\mathcal{H}(\mathcal{V})}$ as the neutral element) on $\mathcal{V}$.
Let $\mathcal{C}$ be a sub differential ring of $\mathcal{A}$ (*i.e.* $\partial_i \mathcal{C} \subset \mathcal{C}, 1 \leq i \leq n$).

For any $f \in \mathcal{A}$, one has $df = (\partial_1 f)dz_1 + \ldots + (\partial_n f)dz_n$. Then [9]

$$\forall S \in \mathcal{A}\langle\langle \mathcal{T}_n \rangle\rangle, \quad \partial_i S = \sum_{w \in \mathcal{T}_n^*} (\partial_i \langle S|w \rangle) w, \quad \mathbf{d}S = \sum_{1 \leq i \leq n} (\partial_i S) dz_i.$$

In $(\mathcal{A}\langle\langle \mathcal{T}_n \rangle\rangle, \partial_1, \ldots, \partial_n)$ ($\mathrm{Const}(\mathcal{A}\langle\langle \mathcal{T}_n \rangle\rangle) = \mathbb{C}\langle\langle \mathcal{T}_n \rangle\rangle$), let us consider

$$(NCDE) \quad \mathbf{d}S = M_n S, \quad \text{where} \quad M_n = \sum_{1 \leq i < j \leq n} \omega_{i,j} t_{i,j} \in \Omega^1(\mathcal{V})\langle \mathcal{T}_n \rangle,$$

where $\omega_{i,j} = d\xi_{i,j}$ is a holomorphic 1-form [10] belonging to $\Omega^1(\mathcal{V})$.

Example ($\xi_{i,j}(z) = \log(z_i - z_j), 1 \leq i < j \leq n$)

Let $\mathcal{C}_0 := \mathbb{C}[\{(\partial_1 \xi_{i,j})^{\pm 1}, \ldots, (\partial_n \xi_{i,j})^{\pm 1}\}_{1 \leq i < j \leq n}]$. Then $\partial_k \mathcal{C}_0 \subset \mathcal{C}_0, 1 \leq k \leq n$.

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9. Recall that $\{\langle S|w \rangle\}_{w \in \mathcal{T}_n^*}$ commute with $\{w\}_{w \in \mathcal{T}_n^*}$.
10. $\mathcal{C} \ni \xi_{i,j}$ is a primitive for $\omega_{i,j}$. It is exact and then is a closed, *i.e.* $d\omega_{i,j} = 0$.

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$$\overbrace{\{x_k\}_{1\le j\le n(n-1)/2}}^{\longleftrightarrow \mathcal{T}_n} \text{ s.t. } \begin{cases} M_n = \sum\limits_{1\le i<j\le n} \omega_{i,j} t_{i,j} = \sum\limits_{k=1}^{n(n-1)/2} F_k x_k = \sum\limits_{l=1}^{n} A_l dz_l, \\ F_k = \sum\limits_{l=1}^{n} f_{l,k} dz_l \quad \text{and then} \quad A_l = \sum\limits_{k=1}^{n(n-1)/2} f_{l,k} x_k. \end{cases}$$

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It follows that $\partial_i A_j - \partial_j A_i + [A_i, A_j] = 0$, for $1 \le i, j \le n$.
Or equivalently[11], $\mathbf{d}M_n - M_n \wedge M_n = 0$.

This induces an ideal, $\mathcal{J}_n$, of relators among $\{t_{i,j}\}_{1\le i<j\le n}$ and then solution can be computed over $\mathcal{H}(\mathcal{V})\langle\langle \mathcal{T}_n \rangle\rangle$ and then over $\mathcal{H}(\mathcal{V})\langle\langle \mathcal{T}_n \rangle\rangle / \mathcal{J}_n$.

$$\mathcal{T}_n = T_n \sqcup \mathcal{T}_{n-1} \longleftrightarrow M_n = \bar{M}_n + M_{n-1}, \text{ where } \bar{M}_n = \sum_{k=1}^{n-1} \omega_{k,n} t_{k,n}.$$

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Iterated integrals and Chen series

Let $\varsigma \rightsquigarrow z$ denotes a path over $\mathcal{V}$ (with fixed endpoints, (ς, z)) :
$\gamma : [0,1] \longrightarrow \mathcal{V}$ s.t. $\gamma(0) = \varsigma = (\varsigma_1, \ldots, \varsigma_n)$ and $\gamma(1) = z = (z_1, \ldots, z_n)$.
The iterated integrals, of $\{\omega_{i,j}\}_{1 \leq i < j \leq n}$ and along $\varsigma \rightsquigarrow z$, are defined by

$$\alpha_\varsigma^z(1_{\mathcal{T}_n^*}) = 1_{\mathcal{H}(\mathcal{V})}, \quad \forall t_{i,j} u \in \mathcal{T}_n^*, \quad \alpha_\varsigma^z(t_{i,j} u) = \int_\varsigma^z \omega_{i,j}(s) \alpha_\varsigma^s(v).$$

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which can be obtained by Picard's iteration [12] :

$$F_0 = 1_{\mathcal{T}_n^*}, \quad \forall i \geq 1, \quad F_i(\varsigma, z) = F_{i-1}(\varsigma, z) + \int_\varsigma^z M_n F_{i-1}(\varsigma, s).$$

Let g be a diffeomorphism on $\mathcal{V}$. The Chen series, of $\{g^* \omega_{i,j}\}_{1 \leq i < j \leq n}$ and along $\varsigma \rightsquigarrow z$, or equivalently, of $\{\omega_{i,j}\}_{1 \leq i < j \leq n}$ and along $g_* \varsigma \rightsquigarrow z$:

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Let g be a diffeomorphism on $\mathcal{V}$. The Chen series, of $\{g^* \omega_{i,j}\}_{1 \leq i < j \leq n}$ and along $\varsigma \rightsquigarrow z$, or equivalently, of $\{\omega_{i,j}\}_{1 \leq i < j \leq n}$ and along $g_* \varsigma \rightsquigarrow z$:

$$C_{g_* \varsigma \rightsquigarrow z} = \sum_{m \geq 0} \sum_{t_{i_1,j_1} \ldots t_{i_m,j_m} \in \mathcal{T}_n^*} \int_\varsigma^z g^* \omega_{i_1,j_1}(s_1) \ldots \int_\varsigma^{s_{m-1}} g^* \omega_{i_m,j_m}(s_m) t_{i_1,j_1} \ldots t_{i_m,j_m}$$

which can be obtained by Picard's iteration with $F_0^* = 1_{\mathcal{T}_n^*}$ and for $i \geq 1$,

$$F_{i-1}^*(\varsigma, z) + \int_\varsigma^z M_n^* F_{i-1}^*(\varsigma, s), \quad \text{where} \quad M_n^* = \sum_{1 \leq i < j \leq n} g^* \omega_{i,j} t_{i,j}.$$

12. which is convergent for the discrete topology.

Sequences of grouplike series for Chen series

Let $\{V_k\}_{k\geq 0}$ and $\{\hat{V}_k\}_{k\geq 0}$ satisfy to the following recursion

$$\forall k \geq 1, \quad F_k(\varsigma, z) = F_0(\varsigma, z) \sum_{t_{i,j}\in\mathcal{T}_{n-1}} \int_{\varsigma}^{z} \omega_{i,j}(s) F_0^{-1}(\varsigma, s) t_{i,j} F_{k-1}(\varsigma, s),$$

with the starting conditions, as being ⧢-grouplike series,

$$V_0(\varsigma, z) = \prod_{l\in\mathcal{L}yn\,T_n} e^{\alpha_\varsigma^z(S_l)P_l} \quad \text{and} \quad \hat{V}_0 = V_0 \mod [\mathcal{L}ie_{\mathcal{A}}\langle\langle T_n\rangle\rangle, \mathcal{L}ie_{\mathcal{A}}\langle\langle T_n\rangle\rangle].$$

$$\sum_{k\geq 0} V_k \quad \text{and} \quad \sum_{k\geq 0} \hat{V}_k \quad \begin{cases} \text{Do they converge ?} \\ \text{What are their limit ?} \end{cases}$$

Let $\varphi_{T_n} := e^{\mathrm{ad}_{-V_0}}$ and $\hat{\varphi}_{T_n} := e^{\mathrm{ad}_{-\hat{V}_0}}$ be the conc-morphisms defined by

$$\varphi_{T_n}^{(\varsigma,z)}(w) = \varphi_{T_n}^{(\varsigma,s_1)}(t_{i_1})\cdots\varphi_{T_n}^{(\varsigma,s_k)}(t_{i_k}), \quad \hat{\varphi}_{T_n}^{(\varsigma,z)}(w) = \hat{\varphi}_{T_n}^{(\varsigma,s_1)}(t_{i_1})\cdots\hat{\varphi}_{T_n}^{(\varsigma,s_k)}(t_{i_k}),$$

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$$\lambda(t\otimes t) = \sum_{v\in T_n^*} a(vt)\otimes r(vt) \quad \text{and} \quad \hat{\lambda}(t\otimes t) = \sum_{v\in T_n^*} a(\hat{v}t)\otimes r(vt).$$

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Volterra expansion like

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H(\varsigma, z) &= 1_{\mathcal{T}_n^*} + \sum_{k\geq 1} \sum_{t_{i_1,j_1}\dots t_{i_k,j_k}\in\mathcal{T}_{n-1}^*} \\
&\quad \int_{\varsigma}^{z} \omega_{i_1,j_1}(s_1)\varphi_{T_n}^{(\varsigma,s_1)}(t_{i_1,j_1})\dots \int_{\varsigma}^{s_{k-1}} \omega_{i_k,j_k}(s_k)\varphi_{T_n}^{(\varsigma,s_k)}(t_{i_k,j_k}), \\
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Theorem

For any $w \in \mathcal{T}_{n-1}^*$, there is $\kappa_w = V_0\varphi_{T_n}(w)$ and $\hat{\kappa}_w = \hat{V}_0\hat{\varphi}_{T_n}(w)$ s.t.

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\sum_{k\geq 0} V_k = V_0 H, \quad \sum_{k\geq 0} \hat{V}_k = \hat{V}_0\hat{H}, \quad C_{\varsigma\rightsquigarrow z} = V_0(\varsigma, z)H(\varsigma, z).
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Using φ_n and $\hat{\varphi}_n$, i.e. reducing by $\mathcal{J}_n$, analogous results hold.

Normalized Chen series as image of diagonal series

$F_\bullet : (\mathbb{C}\langle\mathcal{T}_n\rangle, ⧢, 1_{\mathcal{T}_n^*}) \longrightarrow (\mathcal{H}(\mathcal{V}), \times, 1_{\mathcal{H}(\mathcal{V})})$ is the ⧢-character defined by $F_{1_{\mathcal{T}_n^*}} = 1_{\mathcal{H}(\mathcal{V})}$, for any $t_{i,j} \in \mathcal{T}_n$, $F_{t_{i,j}}(z) = \log(z_i - z_j)$ and

$$\forall w \in \mathcal{L}yn\mathcal{T}_n \setminus \mathcal{T}_n, \quad F_{t_{i,j}w}(z) = \int_0^z \omega_{i,j}(s) F_w(s), \quad \omega_{i,j}(z) = d\log(z_i - z_j).$$

As image by $F_\bullet \otimes \mathrm{Id}$ of $\mathcal{D}_{\mathcal{T}_n}$, the graph of $F_\bullet$ is expressed as follows

Proposition

$$\begin{aligned}
\mathrm{F}_{KZ_n} &= \prod_{l \in \mathcal{L}yn\mathcal{T}_{n-1}}^{\searrow} e^{F_{s_l} P_l} \Bigg(\prod_{\substack{l = l_1 l_2 \\ l_2 \in \mathcal{L}yn\mathcal{T}_{n-1}, l_1 \in \mathcal{L}yn T_n}}^{\searrow} e^{F_{s_l} P_l} \Bigg) \prod_{l \in \mathcal{L}yn T_n}^{\searrow} e^{F_{s_l} P_l} \\
&= \prod_{l \in \mathcal{L}yn T_n}^{\searrow} e^{F_{s_l} P_l} \Bigg(1_{\mathcal{T}_n^*} + \\
&\quad \sum_{k \geq 1} \sum_{\substack{v_1, \ldots, v_k \in T_n^* \\ t_1, \ldots, t_k \in \mathcal{T}_{n-1}}} F_{a(v_1 t_1) \underset{2}{⧢} \ldots \underset{2}{⧢} a(v_k t_k)} r(v_1 t_1) \ldots r(v_k t_k) \Bigg).
\end{aligned}$$

Modulo $[\mathcal{L}ie_{\mathcal{A}}\langle\langle T_n\rangle\rangle, \mathcal{L}ie_{\mathcal{A}}\langle\langle T_n\rangle\rangle]$, *one also has*

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\end{aligned}$$

Solution of KZ_n ($n \geq 4$) by *dévissage*

$$M_n = \bar{M}_n + M_n.$$

For[13] $z_n \to z_{n-1}$, let $s = z_n$ and $s_k = z_n - z_k$ ($1 \leq k \leq n-1$). Then

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Theorem

For $z_n \to z_{n-1}$, $\sqcup\!\sqcup$-grouplike solution of $\mathbf{d}S = M_n S$ can be put in the form $h(z_n)H(z_1, \ldots, z_{n-1})$ such that,

1. *h satisfies $df = N_{n-1} f$. Hence, $h(z_n) \sim_{z_n \to z_{n-1}} (z_{n-1} - z_n)^{t_{n-1,n}}$.*
2. *Letting $(P_{i,j}) : z_i - z_j = 1$, for $i \neq j$, H satisfies $\mathbf{d}S = M_{n-1}^{\varphi_n} S$, where*

$$M_{n-1}^{\varphi_n^{(z^0,z)}}(z) = \sum_{1 \leq i < j \leq n-1} d\log(z_i - z_j)\, \varphi_n^{(z^0,z)}(t_{i,j})$$

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Conversely, for $z_n \to z_{n-1}$, if h satisfies $df = N_{n-1} f$ and H satisfies $\mathbf{d}S = M_{n-1}^{\varphi_n} S$ then $h(z_n)H(z_1, \ldots, z_{n-1})$ is solution of $\mathbf{d}S = M_n S$.

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Solution of KZ_n ($n \geq 4$) by *dévissage*

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Solution of KZ_n ($n \geq 4$) satisfying asymptotic conditions

The previous theorem holds for $z_n \to z_{n-1}$ and can be recursively performed for *dévissage*.
Up to a permutation, it can be adapted for other cases. Hence,

Corollary

In $\mathcal{A}\langle\langle\mathcal{T}_n\rangle\rangle/\mathcal{J}_{\mathcal{R}_n}$, the unique ш-grouplike solution of NCDE, F_{KZ_n}, satisfies

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